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Dual generators of the fundamental group and the moduli space of flat connections

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Abstract

We define the dual of a set of generators of the fundamental group of an oriented 2-surface $S_{g,n}$ of genus g with n punctures and the associated surface $S_{g,n}\setminus D$ with a disc D removed. This dual is another set of generators related to the original generators via an involution and has the properties of a dual graph. In particular, it provides an algebraic prescription for determining the intersection points of a curve representing a general element of the fundamental group $\pi_1(S_{g,n}\setminus D)$ with the representatives of the generators and the order in which these intersection points occur on the generators. We apply this dual to the moduli space of flat connections on $S_{g,n}$ and show that when expressed in terms of both, the holonomies along a set of generators and their duals, the Poisson structure on the moduli space takes a particularly simple form. Using this description of the Poisson structure, we derive explicit expressions for the Poisson brackets of general Wilson loop observables associated with closed, embedded curves on the surface and determine the associated flows on phase space. We demonstrate that the observables constructed from the pairing in the Chern–Simons action generate infinitesimal Dehn twists and show that the mapping class group acts by Poisson isomorphisms.

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1. Introduction

Moduli spaces of flat connections on orientable 2-surfaces arise in many contexts. Our main motivation is their role as the phase space of Chern–Simons gauge theory, in particular the application to the Chern–Simons formulation of (2+1)-dimensional gravity with gauge groups $ISO(2,1)$, $SL(2,\mathbb{C})$, $SL(2,\mathbb{R})\times SL(2,\mathbb{R})/\mathbb{Z}_2$ [1, 2]. Although obtained as quotients of infinite-dimensional spaces of flat connections on the surface, moduli spaces are finite dimensional, which reflects the topological nature of the underlying Chern–Simons theory.

This absence of local degrees of freedom allows one to parametrize them in terms of the holonomies of curves on the surface. This parametrization provides a complete set of gauge-invariant observables, in the following referred to as generalized Wilson loop observables, given as conjugation-invariant functions of the holonomies. Furthermore, the parametrization of the moduli space in terms of holonomies gives rise to an efficient description of its Poisson structure discovered by Fock and Rosly [3]. In Fock and Rosly's formalism, the moduli space is parametrized by the holonomies along a set of generators of the surface's fundamental group and its Poisson structure is given by an auxiliary Poisson structure on an extended phase space which is obtained by associating one copy of the gauge group to each generator.

Due to its simplicity, Fock and Rosly's description of the moduli space has proven useful in the investigation of the phase space of Chern–Simons theory, in particular in the Chern–Simons formulation of (2+1)-dimensional gravity [4, 5]. Moreover, it serves as the starting point for the quantization of Chern–Simons theory. Most quantization approaches such as the combinatorial quantization formalism for Chern–Simons theory with compact, semisimple gauge groups [6–8] and the related approaches in [9, 10] for the case of, respectively, gauge group $SL(2,\mathbb{C})$ and $G \ltimes \mathfrak{g}^*$ are based on Fock and Rosly's description of the moduli space and take the holonomies along a set of generators of the fundamental group as their basic variables.

The drawback of this description is that it obscures the geometrical nature of the theory and thereby complicates its physical interpretation. For instance, it is well known that the Poisson bracket of Wilson loop observables associated with closed curves on the surface is determined by the intersection behaviour of these curves, i.e. the number of intersection points and the associated oriented intersection numbers. It is shown in [5] for the Chern–Simons formulation of (2+1)-dimensional gravity with vanishing cosmological constant that this dependence on intersection points is crucial for the geometrical interpretation of the observables and the associated phase space transformations they generate via the Poisson bracket. However, in Fock and Rosly's formalism [3] based on the holonomies along a set of generators of the fundamental group, this dependence on intersection points is not directly apparent. The main reason is the lack of a direct link between the expressions of elements of the fundamental group in terms of the generators and the intersection points of their representatives on the surface. Given the expression of an element of the fundamental group in terms of the generators, it is in general difficult to determine how its representative intersects the representatives of the generators without explicitly drawing these curves on the surface. This difficulty manifests itself in Fock and Rosly's description of the Poisson structure, where one uses these expressions in terms of the generators to calculate the Poisson bracket of the associated Wilson loop observables.

This problem and its relevance for the physical applications are the main motivation of the present paper. We consider oriented 2-surfaces $S_{g,n}$ of general genus g and with n punctures and the associated surfaces $S_{g,n}\setminus D$ with a disc D removed and introduce the concept of a dual for a set of generators of the fundamental groups $\pi_1(S_{g,n})$, $\pi_1(S_{g,n}\setminus D)$. This dual is another set of generators related to the original generators via an involution and has the properties of a dual graph. It allows us to keep track of the intersection points of general curves on the surface with representatives of the generators. More precisely, we show that for any element of the fundamental group $\pi_1(S_{g,n}\setminus D)$, the intersection points of a representing curve with the representatives of the generators of $\pi_1(S_{g,n}\setminus D)$ are labelled by the factors in its expression as a product in the associated dual generators and their inverses. Moreover, for elements with embedded representatives, this expression allows one to determine algebraically the order in which these intersection points occur on the generators.

We then apply this duality for the fundamental group to Fock and Rosly's description [3] of the moduli space of flat connections. We find that, when expressed in terms of both, the holonomies along the original set of generators of the fundamental group and those along their duals, the Poisson structure takes a particularly simple form in which its dependence on intersection points is encoded algebraically and readily apparent. This allows us to derive explicit expressions for the Poisson bracket of the generalized Wilson loop observables associated with embedded curves on the surface and to determine the flows these observables generate via the Poisson bracket. In particular, we consider the Wilson loop observables constructed from the pairing in the Chern–Simons action and show that the flows generated by these observables have the interpretation of infinitesimal Dehn twists. We thus give an independent re-derivation of Goldman's classic results in [11] and generalize these results to surfaces with punctures. However, while the results in [11] are presented in a more abstract and geometrical language, the formulation in this paper is purely algebraic. As it gives explicit expressions for the Poisson brackets of Wilson loop observables and the associated phase space transformations in terms of the holonomies along a set of generators of the fundamental group, our formulation provides a direct link with the description of the phase space and the quantization formalisms in [4, 6–10]. This may prove useful in the investigation of the associated observables and transformation in quantized Chern–Simons theory. Moreover, in the Chern–Simons formulation of (2+1)-dimensional gravity the explicit parametrization in terms of holonomies allows one to establish a link with the geometrical formalism and to relate these flows to the geometrical construction of (2+1)-spacetimes via grafting [12].

The paper is structured as follows.

In section 2, we motivate and define the concept of a dual for a set of generators of the fundamental groups $\pi_1(S_{g,n})$, $\pi_1(S_{g,n}\setminus D)$ and investigate the involution which maps a set of generators to its dual. We show that the intersection points of a general curve on the surface $S_{g,n}\setminus D$ with the representatives of the generators are labelled by the factors in the expression of its homotopy equivalence class in terms of their duals.

In section 3, we investigate the combinatorial and geometrical properties of the dual generators. For elements of $\pi_1(S_{g,n}\setminus D)$ with embedded representatives, we show that their expression in terms of the dual generators allows one to algebraically determine the order in which these intersection points occur on the representatives of the generators. Furthermore, we demonstrate that the involution defines an (almost) unique assignment of these intersection points between the different factors in the expression of this element as a product in the original generators. We show how this assignment of intersection points corresponds to a graphical decomposition of the associated curve on $S_{g,n}\setminus D$ into representatives of the generators.

In section 4, we apply the dual generators to the description of the moduli space of flat connections on $S_{g,n}$ in terms of the holonomies of a set of generators of the fundamental group. We summarize the relevant facts about Chern–Simons theory and Fock and Rosly's description [3] of the moduli space of flat connections on $S_{g,n}$. We show that Fock and Rosly's auxiliary Poisson structure can be cast in a particularly simple form when expressed in terms of both the holonomies along the original set of generators and their duals, and discuss how this reflects its dependence on intersection points. Finally, we determine the transformation of this Poisson structure under the involution that maps the original generators to their duals.

In section 5, we use this description of the Poisson structure to determine the Poisson brackets of the generalized Wilson loop observables associated with elements of $\lambda \in \pi_1(S_{g,n}\setminus D)$ and conjugation-invariant functions on the gauge group. For Wilson loop observables associated with elements with embedded representatives, we derive the flows on phase space these observables generate via the Poisson bracket. By using both, the graphical assignment of intersection points and the ordering algorithm given in section 3, we then obtain

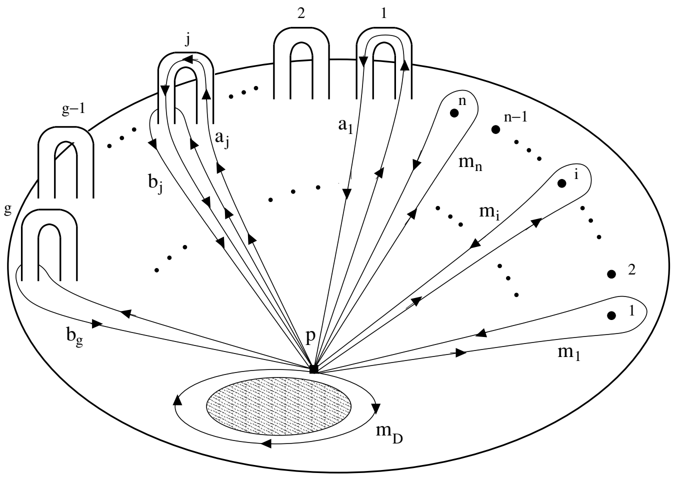

Figure 1. The generators of the fundamental group $\pi_1(S_{g,n}\backslash D)$.

explicit expressions for the transformation of the holonomies along our set of generators in terms of the expression of λ as a product in the generators of $\pi_1(S_{g,n}\backslash D)$ and their duals.

In section 6, we consider a set of generic Wilson loop observables associated with the Ad-invariant symmetric bilinear form in the Chern–Simons action. We show that the flows generated by these observables represent infinitesimal Dehn twists and that the mapping class group $\mathrm{Map}(S_{g,n}\backslash D)$ acts by Poisson isomorphisms. We then use this identity to determine the transformation of Fock and Rosly's Poisson structure under a general automorphism of the fundamental group $\pi_1(S_{g,n}\backslash D)$ which acts on the punctures by conjugation and maps a curve around the disc to itself or its inverse.

Section 7 contains our outlook and conclusions, and in the appendix we list a set of generators for the mapping class groups $\mathrm{Map}(S_{g,n}\backslash D)$, $\mathrm{Map}(S_{g,n})$.

2. The dual of a set of generators of the fundamental group

In this paper, we consider orientable 2-surfaces $S_{g,n}$ of genus g and with n punctures and the associated surfaces $S_{g,n}\backslash D$ obtained from $S_{g,n}$ by removing a disc D. Both the fundamental groups $\pi_1(S_{g,n})$ and $\pi_1(S_{g,n}\backslash D)$ are generated by the homotopy equivalence classes of a set of loops $m_i, i = 1, \ldots, n$, around each puncture and two curves $a_j, b_j, j = 1, \ldots, g$, for each handle as shown in figure 1. While the fundamental group $\pi_1(S_{g,n}\backslash D)$ is the free group generated by the homotopy equivalence classes of m_i, a_j, b_j

$$\pi_1(S_{g,n}\backslash D) = \langle m_1, \ldots, m_n, a_1, b_1, \ldots, a_g, b_g\rangle, \tag{1}$$

the fundamental group $\pi_1(S_{g,n})$ is obtained by imposing a single defining relation

$$\pi_1(S_{g,n}) = \langle m_1, \ldots, m_n, a_1, b_1, \ldots, a_g, b_g; \left[b_g, a_g^{-1}\right] \circ \ldots \circ \left[b_1, a_1^{-1}\right] \circ m_n \circ \ldots \circ m_1 = 1\rangle$$
$$\left[b_i, a_i^{-1}\right] = b_i \circ a_i^{-1} \circ b_i^{-1} \circ a_i, \tag{2}$$

which amounts to the requirement that the loop around the disc D representing

$$m_D = \left[b_g, a_g^{-1}\right] \circ \ldots \circ \left[b_1, a_1^{-1}\right] \circ m_n \circ \ldots \circ m_1 \tag{3}$$

is contractible. Throughout the paper, we work with a fixed set of generators as depicted in figure 1 and representatives based at a fixed point $p \in S_{g,n}$, $p \in S_{g,n}\backslash D$, which all homotopies keep fixed. In the following we will often not mention the dependence on the basepoint explicitly and do not distinguish notationally between curves on the surfaces $S_{g,n}$, $S_{g,n}\setminus D$ and their equivalence classes in the fundamental groups $\pi_1(S_{g,n})$, $\pi_1(S_{g,n}\setminus D)$. We will also denote by the same letter elements of the fundamental group $\pi_1(S_{g,n}\setminus D)$ and the corresponding elements of $\pi_1(S_{g,n})$ obtained via the canonical map $\pi_1(S_{g,n}\setminus D) \to \pi_1(S_{g,n})$.

To define the dual of a set of generators of the fundamental groups $\pi_1(S_{g,n})$, $\pi_1(S_{g,n}\setminus D)$, we first establish the desired properties of this dual and then address its existence and uniqueness. Heuristically, the dual of a set of generators $\{m_1, \ldots, m_n, a_1, b_1, \ldots, a_g, b_g\}$ of the fundamental groups $\pi_1(S_{g,n})$, $\pi_1(S_{g,n}\setminus D)$ should be another set of generators $\{\overline{m}_1, \ldots, \overline{m}_n, \overline{a}_1, \overline{b}_1, \ldots, \overline{a}_g, \overline{b}_g\}$ related to the original generators by an automorphism $I \in \mathrm{Aut}(\pi_1(S_{g,n}))$, $I \in \mathrm{Aut}(\pi_1(S_{g,n}\setminus D))$ and with the following properties:

(1) The dual of the dual set of generators should be the original set of generators, i.e. the automorphism I should be an involution $I^2 = 1$.

(2) The dual set of generators should be geometrically equivalent to our original set of generators, i.e. the representatives of the original and the dual generators should be related by homeomorphisms of $S_{g,n}$, $S_{g,n}\setminus D$. In other words, we require an involution $I \in \mathrm{Aut}(S_{g,n}\setminus D)$ which is induced by a homeomorphism of $S_{g,n}\setminus D$ which fixes the punctures as a set and the boundary of the disc and which induces an involution of $\pi_1(S_{g,n})$ associated with a homeomorphism of $S_{g,n}$ which fixes the punctures as a set. Results from combinatorial group theory and geometric topology (see for instance theorems 3.4.5, 3.4.6, 3.4.7 in [13]) imply that is the case if and only if the automorphism $I \in \mathrm{Aut}(S_{g,n}\setminus D)$ satisfies

$$\begin{aligned} &I(m_i) = w_i m_{\sigma(i)}^{\epsilon_i} w_i^{-1} \qquad I(m_D) = w m_D^{\epsilon} w^{-1} \\ &\omega(w_1)\epsilon_1 = \cdots = \omega(w_n)\epsilon_n = \omega(w)\epsilon \in \{\pm 1\} \end{aligned} \tag{4}$$

where $w_i, w \in \pi_1(S_{g,n}\setminus D)$, $\epsilon, \epsilon_i \in \{\pm 1\}$, $\sigma \in S_n$ is a permutation of the punctures and $\omega(x) = 1$ if $x \in \pi_1(S_{g,n}\setminus D)$ corresponds to a separating curve and $\omega(x) = -1$ otherwise. By applying an inner automorphism of $\pi_1(S_{g,n}\setminus D)$ which conjugates all elements with a fixed element $\lambda \in \pi_1(S_{g,n}\setminus D)$, we can set $w = 1$ in (4), which we will assume in the following. The corresponding homeomorphism is orientation preserving and orientation reversing, respectively, if $\epsilon = 1$ and $\epsilon = -1$.

(3) Finally, we require that the dual of our set of generators of the fundamental group should have properties similar to those of a dual graph and should allow one to keep track of the intersection points of general curves on the surface with our set of generators m_i, a_j, b_j. The intersection points of a general curve on the surface with the representatives of the generators m_i, a_j, b_j should be labelled by the factors in the expression of its homotopy equivalence class in terms of the dual generators. As the expression of a general element in terms of the dual generators is unique only for the fundamental group $\pi_1(S_{g,n}\setminus D)$, we impose this condition on the involution $I \in \mathrm{Aut}(\pi_1(S_{g,n}\setminus D))$. More precisely, for each element $\lambda \in \pi_1(S_{g,n}\setminus D)$ we consider the unique expression of λ as a *reduced word* in the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$

$$\lambda = \overline{x}_r^{\alpha_r} \cdots \overline{x}_1^{\alpha_1} = I\left(x_r^{\alpha_r} \cdots x_1^{\alpha_1}\right) \qquad x_k \in \{m_1, \ldots, b_g\}, \quad \overline{x}_k = I(x_k), \quad \alpha_k \in \{\pm 1\} \tag{5}$$

with $\overline{x}_k^{\alpha_k} \neq \overline{x}_{k+1}^{-\alpha_{k+1}}$ for $k = 1, \ldots, r-1$ and require that the intersection points of a curve representing λ with the representatives of the generators a_j, b_j are in one-to-one correspondence with factors $\overline{x}_k = \overline{a}_j, \overline{x}_k = \overline{b}_j$ in (5). For the generators m_i associated with the punctures, we require that each factor $\overline{x}_k = \overline{m}_i$ in (5) corresponds to a pair of intersection points of this curve with a representative of m_i.

Together the first and the second requirement determine the involution $I \in \mathrm{Aut}(\pi_1(S_{g,n}\backslash D))$ up to composition $I \mapsto \rho \circ I$ with an automorphism $\rho \in \mathrm{Aut}(\pi_1(S_{g,n}\backslash D))$ which satisfies (4) with $w = 1$ and the additional condition

$$\rho^{-1} = I\rho I. \tag{6}$$

The third requirement defines I up to conjugation with an automorphism $\rho \in \mathrm{Aut}(\pi_1(S_{g,n}\backslash D))$,

$$I \mapsto \rho \circ I \circ \rho^{-1}, \tag{7}$$

which satisfies (4) with $w = 1$, since the intersection points of $\rho(x)$, $x \in \{m_1, \ldots, b_g\}$, with $\rho(\lambda)$ correspond one to one to intersection points of x with λ and are labelled by the factors in the expression of $\rho(\lambda)$ as a reduced word in $\rho \circ I \circ \rho^{-1}(m_i)$, $\rho \circ I \circ \rho^{-1}(a_j)$, $\rho \circ I \circ \rho^{-1}(b_j)$. Furthermore, all involutions conjugated to a given involution I in that way are obtained from automorphisms $\rho \in \mathrm{Aut}(\pi_1(S_{g,n}\backslash D))$ which satisfy (4) with $\epsilon = 1$. We will discuss in section 6 that such automorphisms of $\pi_1(S_{g,n}\backslash D)$ represent elements of the mapping class group $\mathrm{Map}(S_{g,n}\backslash D)$. Hence, two involutions satisfying the requirements above and related by conjugation with such an automorphism correspond to the choice of an alternative set of generators $\{\rho(m_1), \ldots, \rho(b_g)\}$, and we have the following theorem.

Theorem 2.1. *The requirements above determine the involution $I \in Aut(\pi_1(S_{g,n}\backslash D))$ uniquely up to the initial choice of the generators of $\pi_1(S_{g,n}\backslash D)$.*

After formulating our concept of the dual of a set of generators of the fundamental group $\pi_1(S_{g,n})$, $\pi_1(S_{g,n}\backslash D)$ and discussing its uniqueness, we will now demonstrate that such a set of generators with the required properties exists. We define an automorphism $I \in \mathrm{Aut}(\pi_1(S_{g,n}\backslash D))$ explicitly by its action on the generators m_i, a_j, b_j and then verify that it satisfies the above requirements.

Lemma 2.2. *Let $I \in Aut(S_{g,n}\backslash D)$ be defined by its action on our set of generators*

$$\begin{aligned}
&I(m_i) = \overline{m}_i = m_1^{-1} \circ \cdots \circ m_{i-1}^{-1} \circ m_i^{-1} \circ m_{i-1} \cdots \circ m_1\\
&I(a_j) = \overline{a}_j = m_1^{-1} \circ \cdots \circ m_n^{-1} \circ h_1^{-1} \circ \cdots \circ h_{j-1}^{-1} h_j^{-1} \circ b_j \circ h_{j-1} \circ \cdots \circ h_1 \circ m_n \circ \cdots \circ m_1\\
&I(b_j) = \overline{b}_j = m_1^{-1} \circ \cdots \circ m_n^{-1} \circ h_1^{-1} \circ \cdots \circ h_{j-1}^{-1} \circ h_j^{-1} \circ a_j \circ h_{j-1} \circ \cdots \circ h_1 \circ m_n \circ \cdots \circ m_1\\
&\textit{with} \quad h_j = [b_j, a_j^{-1}] = b_j \circ a_j^{-1} \circ b_j \circ a_j.
\end{aligned} \tag{8}$$

Then, I is an involution and satisfies the requirements (4) with $w = 1, \epsilon = -1$. It therefore arises from an orientation-reversing homeomorphism of $S_{g,n}\backslash D$ and induces an automorphism of $\pi_1(S_{g,n})$ which arises from an orientation-reversing homeomorphism of $S_{g,n}$.

It remains to show that the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$ defined by (8) are dual to our original generators m_i, a_j, b_j in a geometrical sense, i.e. that they satisfy the third requirement and allow us to determine the intersection points of a general element $\lambda \in \pi_1(S_{g,n}\backslash D)$ with the generators m_i, a_j, b_j. For this, we consider a set of representing curves on $S_{g,n}\backslash D$ as depicted in figure 2, and note that the curves representing the generators $\overline{a}_j, \overline{b}_j$, respectively, intersect only a_j and b_j, in a single point. Similarly, the representatives of the generators $\overline{m}_i$ intersect only m_i, but in two points and with opposite oriented intersection numbers.

To explore the geometric properties of the dual generators further, we cut the surface $S_{g,n}\backslash D$ along a set of curves representing the generators $m_i, a_j, b_j \in \pi_1(S_{g,n}\backslash D)$ as shown in figure 3. We then obtain n punctured discs D_i and a simply connected $(4g + n + 1)$-gon $P_{g,n}^D$ depicted in figure 4. Each of the corners x_i, $i = 0, \ldots, n + 4g$, of the polygon $P_{g,n}^D$ corresponds

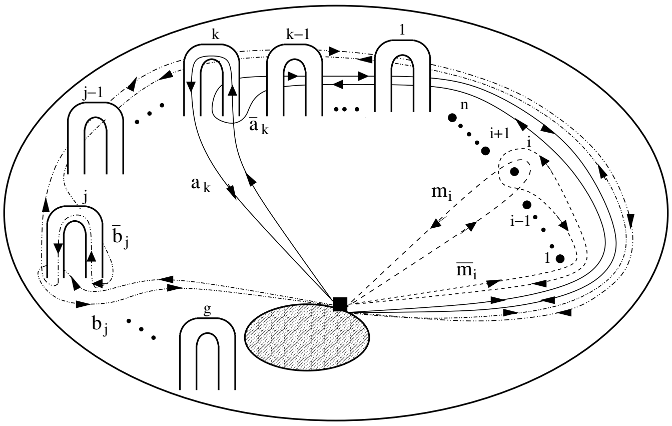

Figure 2. The generators and dual generators of the fundamental group $\pi_1(S_{g,n}\setminus D)$.

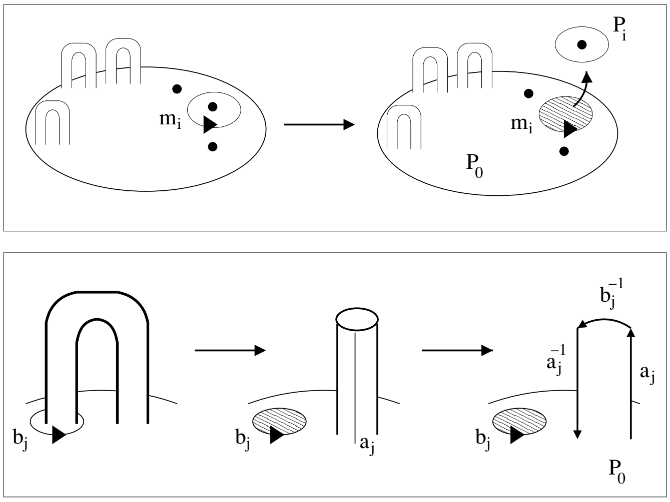

Figure 3. Cutting the surface $S_{g,n}\setminus D$ along the generators of the fundamental group $\pi_1(S_{g,n}\setminus D)$.

to the basepoint $p \in S_{g,n}\setminus D$. The side between x_0 and x_{n+4g} represents the boundary of the disc D, the sides between x_{i-1} and x_i, $i = 1, \dots, n$, the generators m_i and the remaining $4g$ sides a_j, a'_j, b_j, b'_j correspond pairwise to the generators a_j, b_j, $j = 1, \dots, g$. In the following it will be useful to represent the generators m_i by a curve obtained by composing an arc v_i from the basepoint $p \in S_{g,n}\setminus D$ to the ith puncture and an infinitesimal circle c_i around the puncture as shown in figure 5. This yields a subdivision of the side m_i into three segments v_i, v'_i, c_i as shown in figure 4.

We now consider the oriented segments x_0x_k from the corner x_0 to corners x_k, $k = 1, \dots, n+4g$ of the polygon $P^D_{g,n}$. Each of these segments represents a certain element of the fundamental group $\pi_1(S_{g,n}\setminus D)$ which is given in terms of the generators m_i, a_j, b_j and their

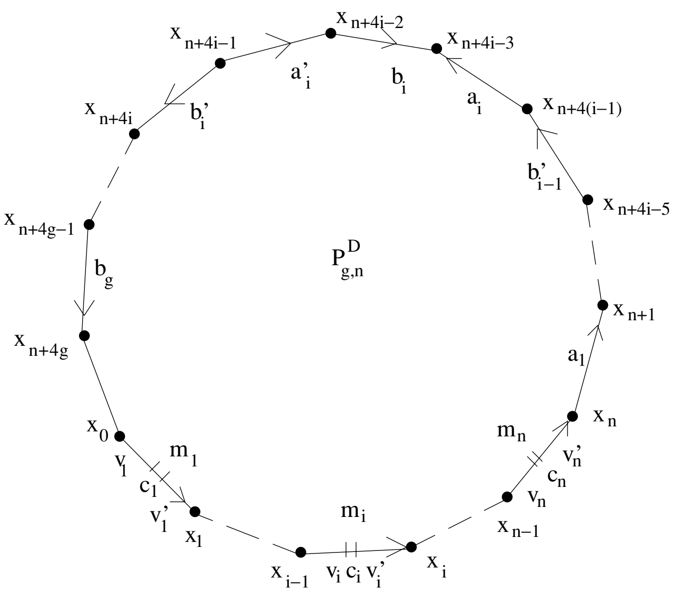

Figure 4. The polygon $P^D_{g,n}$.

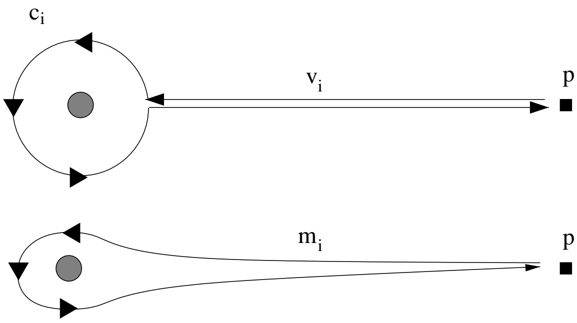

Figure 5. The decomposition of a generator m_i into an arc v_i and an infinitesimal circle c_i.

duals by

$$
\begin{aligned}
x_0x_i &\cong m_i \circ \cdots \circ m_1 = \overline{m}_1^{-1} \circ \cdots \circ \overline{m}_i^{-1} \qquad \text{for} \quad 1 \leqslant i \leqslant n \\
x_0x_{n+4j-3} &\cong a_j \circ h_{j-1} \circ \cdots \circ h_1 \circ m_n \circ \cdots \circ m_1 \\
&= \overline{m}_1^{-1} \circ \cdots \circ \overline{m}_n^{-1} \circ \overline{h}_1^{-1} \circ \cdots \circ \overline{h}_{j-1} \circ \overline{a}_j^{-1} \circ \overline{b}_j \circ \overline{a}_j \\
x_0x_{n+4j-2} &\cong b_j^{-1} \circ a_j \circ h_{j-1} \circ \cdots \circ h_1 \circ m_n \circ \cdots \circ m_1 \\
&= \overline{m}_1^{-1} \circ \cdots \circ \overline{m}_n^{-1} \circ \overline{h}_1^{-1} \circ \cdots \circ \overline{h}_{j-1} \circ \overline{a}_j^{-1} \circ \overline{b}_j \\
x_0x_{n+4j-1} &\cong a_j^{-1} \circ b_j^{-1} \circ a_j \circ h_{j-1} \circ \cdots \circ h_1 \circ m_n \circ \cdots \circ m_1 \\
&= \overline{m}_1^{-1} \circ \cdots \circ \overline{m}_n^{-1} \circ \overline{h}_1^{-1} \circ \cdots \circ \overline{h}_{j-1} \circ \overline{a}_j^{-1} \\
x_0x_{n+4j} &\cong h_j \circ h_{j-1} \circ \cdots \circ h_1 \circ m_n \circ \cdots \circ m_1 = \overline{m}_1^{-1} \circ \cdots \circ \overline{m}_n^{-1} \circ \overline{h}_1^{-1} \circ \cdots \circ \overline{h}_j \\
&\quad \text{for} \quad 1 \leqslant j \leqslant g.
\end{aligned}
\tag{9}
$$

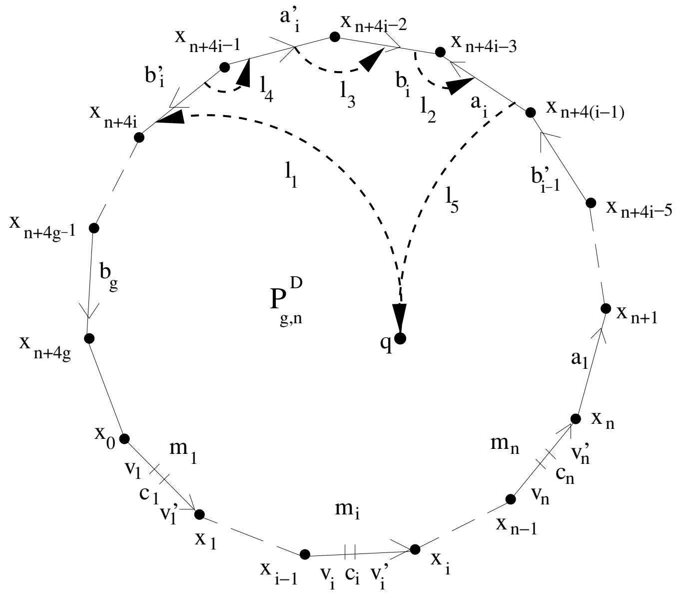

Figure 6. A curve representing $\lambda = \overline{a}_i^{-1} \circ \overline{b}_i \circ \overline{a}_i \circ \overline{b}_i^{-1}$ on $P_{g,n}^D$.

Both the generators m_i, a_j, b_j and their duals $\overline{m}_i, \overline{a}_j, \overline{b}_j$ are obtained by composing two segments $x_0 x_k$. For the generators associated with the punctures this representation is unique:

$$m_i = (x_0 x_i) \circ (x_0 x_{i-1})^{-1} \qquad \overline{m}_i = (x_0 x_i)^{-1} \circ (x_0 x_{i-1}), \tag{10}$$

while there are two possibilities for each generator $a_j, b_j, \overline{a}_j, \overline{b}_j$:

$$\begin{aligned} a_j &= (x_0 x_{n+4j-3}) \circ (x_0 x_{n+4j-4})^{-1} = (x_0 x_{n+4j-2}) \circ (x_0 x_{n+4j-1})^{-1} \\ \overline{a}_j &= (x_0 x_{n+4j-1})^{-1} \circ (x_0 x_{n+4j-4}) = (x_0 x_{n+4j-2})^{-1} \circ (x_0 x_{n+4j-3}) \end{aligned} \tag{11}$$

$$\begin{aligned} b_j &= (x_0 x_{n+4j-3}) \circ (x_0 x_{n+4j-2})^{-1} = (x_0 x_{n+4j}) \circ (x_0 x_{n+4j-3})^{-1} \\ \overline{b}_j &= (x_0 x_{n+4j-1})^{-1} \circ (x_0 x_{n+4j-2}) = (x_0 x_{n+4j})^{-1} \circ (x_0 x_{n+4j-3}). \end{aligned} \tag{12}$$

To demonstrate that the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$ allow us to determine the intersection points of general embedded curves on $S_{g,n} \backslash D$ with the generators m_i, a_j, b_j, we consider an embedded curve $c : [0, 1] \to S_{g,n} \backslash D$, $c(0) = c_\lambda(1) = q$ which does not contain the basepoint $p \in S_{g,n} \backslash D$. Furthermore, we require that c has a minimum number of intersection points with the representatives of the generators m_i, a_j, b_j, i.e. that the number of intersection points cannot be reduced by applying a homotopy which fixes the basepoint.

After the surface $S_{g,n} \backslash D$ is cut along the representatives of the generators m_i, a_j, b_j, the curve c gives rise to a set of oriented segments l_i, $i = 1, \ldots, r+1$, on the polygon $P_{g,n}^D$ as shown in figure 6. We denote the starting and endpoints of the segments l_i by, respectively, s_i and t_i. With the exception of the starting point of the first and the endpoint of the last segment $s_1 = t_r = q$, all other starting and endpoints lie on the sides $m_i, a_j, a'_j, b_j, b'_j$ of $P_{g,n}^D$. Without changing the homotopy equivalence class of c, we can ensure that all intersection points with the sides of $P_{g,n}^D$ representing the generators m_i occur on v_i or v'_i. For segments l_k which end on v_i or v'_i, the next segment l_{k+1} then starts at the corresponding point on v'_i and v_i, respectively. Similarly, for segments that end in a point on a side a_j or b_j, the next segment then starts in the corresponding point on a'_j or b'_j and vice versa.

We can now move the starting and endpoints of the segments l_k towards the corners of the polygon $P^D_{g,n}$. For each segment ending on a side m_i, we have $t_k = x_{i-1}$, $s_k = x_i$ if $t_k \in v_i$ and $t_k = x_i$, $s_{k+1} = x_{i-1}$ if $t_k \in v'_i$. For a segment l_k ending in a side a_j, we can either move its endpoint t_k to the starting point of a_j and hence the starting point s_{k+1} to the starting point of a'_j or move t_k and s_{k+1} to the endpoints of, respectively, a_j and a'_j. The first possibility yields $t_k = x_{n+4j-4}$, $s_{k+1} = x_{n+4j-1}$, the second $t_k = x_{n+4j-3}$, $s_{k+1} = x_{n+4j-2}$, and the corresponding expressions for a segment ending on a'_j are given by exchanging t_k and s_{k+1}. Similarly, for a segment l_k ending on a side b_j, we can move its endpoint t_k and the starting point s_{k+1} to either the starting points of b_j and b'_j, which implies $t_k = x_{n+4j-2}$, $s_{k+1} = x_{n+4j-1}$ or to their endpoints which yields $t_k = x_{n+4j-1}$, $s_{k+1} = x_{n+4j}$. Using the decomposition (10), (11), (12) of the dual generators in terms of the segments connecting the basepoint with the corners of the polygon $P^D_{g,n}$, we then obtain

$$(x_0 s_{k+1})^{-1}(x_0 t_k) = \begin{cases} \overline{m}_i & \text{if} \quad t_k \in v_i, i = 1, \ldots, n \\ \overline{m}_i^{-1} & \text{if} \quad t_k \in v'_i, i = 1, \ldots, n \\ \overline{a}_j & \text{if} \quad t_k \in a_j, j = 1, \ldots, g \\ \overline{a}_j^{-1} & \text{if} \quad t_k \in a'_j, j = 1, \ldots, g \\ \overline{b}_j & \text{if} \quad t_k \in b_j, j = 1, \ldots, g \\ \overline{b}_j^{-1} & \text{if} \quad t_k \in b'_j, j = 1, \ldots, g. \end{cases} \tag{13}$$

Note that the two possibilities of moving points on sides a_j, a'_j, b_j, b'_j to either the starting point or to the endpoint of the sides a_j, a'_j, b_j, b'_j correspond to the two different expressions for $\overline{a}_j, \overline{b}_j$ in (11), (12).

By expressing the segments l_k on the polygon $P^D_{g,n}$ in terms of the segments $x_0 s_k, x_0 t_k$, we find that the homotopy equivalence class of c is given by

$$\begin{aligned} c &= l_{r+1} \circ l_{r-1} \circ \cdots \circ l_1 \\ &= (x_0 q) \circ (x_0 s_{r+1})^{-1}(x_0 t_r) \circ (x_0 s_r)^{-1}(x_0 t_{r-1}) \circ \cdots \circ (x_0 s_2)^{-1}(x_0 t_1) \circ (q x_0), \end{aligned} \tag{14}$$

where $x_0 q$ stands for a segment connecting the point q with the basepoint x_0. By inserting identity (13) into this expression we then obtain the unique expression of the homotopy equivalence class of c as a reduced word in the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$. As the starting and endpoints of segments l_k correspond to intersection points of c with the generators m_i, a_j, b_j, this implies that the intersection points of c with the representatives of a_j, b_j are in one-to-one correspondence with factors $\overline{a}_j, \overline{b}_j$ in the expression of the homotopy equivalence class of c as a product in the dual generators $\overline{a}_j, \overline{b}_j$. Similarly, each factor $\overline{m}_i$ corresponds to a pair of intersection points of c with a representative of m_i. Furthermore, if we define the oriented intersection number $\epsilon(\lambda, \eta)$ of two curves $\lambda, \eta \in \pi_1(S_{g,n}\backslash D)$ to be positive if η crosses λ from the left to the right in the direction of λ, we find that the exponents of $\overline{m}_i, \overline{a}_j, \overline{b}_j$ in (13) determine the oriented intersection numbers associated with these intersection points. We obtain the following theorem.

Theorem 2.3. *Consider an element $\lambda \in \pi_1(q, S_{g,n}\backslash D) \cong \pi_1(p, S_{g,n}\backslash D)$ with an embedded representative which intersects the generators m_i, a_j, b_j in a minimum number of points. Express λ as a reduced word in the generators: $\overline{m}_i, \overline{a}_j, \overline{b}_j \in \pi_1(S_{g,n}\backslash D)$*

$$\lambda = \overline{x}_r^{\alpha_r} \cdots \overline{x}_1^{\alpha_1} \qquad \overline{x}_k \in \{\overline{m}_1, \ldots, \overline{m}_n, \overline{a}_1, \overline{b}_1, \ldots, \overline{a}_g, \overline{b}_g\}, \quad \alpha_k \in \{\pm 1\}. \tag{15}$$

Then, the intersection points of the curve representing λ with the generators a_j, b_j are in one-to-one correspondence with factors $\overline{x}_k = a_j, \overline{x}_k = b_j$ in the expression (15). The associated exponents α_k determine the oriented intersection number $\alpha_k = \epsilon(a_j, \lambda)$ for $\overline{x}_k = \overline{a}_j$, $\alpha_k = -\epsilon(b_j, \lambda)$ for $\overline{x}_k = \overline{b}_j$. Similarly, each factor $\overline{x}_k = \overline{m}_i$ in (15) corresponds to a pair of intersection points of λ with the generator m_i with opposite intersection numbers, and the exponent α_k gives the oriented intersection number of the intersection point which occurs first on m_i.

The expression of an element $\lambda \in \pi_1(S_{g,n}\setminus D)$ as a reduced word in the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$ therefore determines the intersection points of its representatives with the representatives of the generators m_i, a_j, b_j and the associated oriented intersection numbers. Note that since we require that homotopy does not move the basepoint, each curve on $S_{g,n}\setminus D$ corresponds to an element $\lambda \in \pi_1(S_{g,n}\setminus D)$ and not a conjugacy class $[\lambda] = \{\tau\lambda\tau^{-1} | \tau \in \pi_1(S_{g,n}\setminus D)\}$. Different elements of a conjugacy class $[\lambda]$ therefore have different numbers of intersection points with the generators m_i, a_j, b_j. In particular, we find that this number is minimal for those elements of $[\lambda]$ which are represented by a cyclically reduced word in the generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$, i.e. expressions of the form (15) with $\overline{x}_r^{\alpha_r} \neq \overline{x}_1^{-\alpha_1}$.

3. The algebraic properties of the dual generators

3.1. The polygon picture: ordering the intersection points

After showing how the expression of a general element $\lambda \in \pi_1(S_{g,n}\setminus D)$ as a reduced word in the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$ determines the number of its intersection points with the generators m_i, a_j, b_j and the oriented intersection numbers, we will now demonstrate that for elements with embedded representatives it also determines the order in which these intersection points occur on each generator m_i, a_j, b_j. Moreover, we show that the dual generators allow one to assign these intersection points (almost) uniquely between the different factors in the expression of λ as a reduced word in the generators m_i, a_j, b_j and that this assignment of the intersection points can be implemented via a graphical procedure.

In the following we assume that $\lambda \in \pi_1(S_{g,n}\setminus D)$ has an embedded representative and that its expression (15) as a reduced word in $\overline{m}_i, \overline{a}_j, \overline{b}_j$ is also cyclically reduced $\overline{x}_r^{\alpha_r} \neq \overline{x}_1^{-\alpha_1}$. To determine the order in which the intersection points of λ with the generators m_i, a_j, b_j occur on each generator, we determine the order of the associated points on the sides of the polygon $P^D_{g,n}$. For this, we recall the discussion from the last section, where it was shown that after cutting the surface $S_{g,n}\setminus D$ along the generators m_i, a_j, b_j a curve representing λ gives rise to a set of segments on the polygon $P^D_{g,n}$. Each factor $\overline{x}_k^{\alpha_k}$ in the expression (15) of λ as a cyclically reduced word in the dual generators corresponds to two intersection points of λ with the sides of polygon $P^D_{g,n}$. One of these intersection points is realized as the endpoint t_k of a segment l_k and the other one as the starting point s_{k+1} of the next segment l_{k+1}, and the factor $x_k^{\alpha_k}$ identifies these points. This implies that intersection points of a representative of λ with the sides of the polygon $P^D_{g,n}$ are in one-to-one correspondence with cyclic permutations of λ and its inverse. This allows us to identify the intersection points $\{t_1, s_2, t_2, \ldots, s_r, t_r, s_{r+1}\}$ on polygon $P^D_{g,n}$ with elements of the set of cyclic permutations

$$\mathrm{CPerm}(\lambda) = \{\lambda_k | k = 1, \ldots, r\} \cup \{(\lambda_k)^{-1} | k = 1, \ldots, r\} \tag{16}$$

$$\text{where} \quad \lambda_k = \overline{x}_k^{\alpha_k} \cdots \overline{x}_1^{\alpha_1} \overline{x}_r^{\alpha_r} \cdots \overline{x}_{k+1}^{\alpha_{k+1}}, \qquad k = 1, \ldots, r, \tag{17}$$

by setting

$$\begin{aligned} &P : \{t_1, s_2, t_2, \ldots, s_r, t_r, s_{r+1}\} \to \mathrm{CPerm}(\lambda) \\ &P(t_k) = \lambda_{k-1} \quad P(s_{k+1}) = \lambda_k^{-1} \qquad k = 1, \ldots, r. \end{aligned} \tag{18}$$

We find that an intersection point $p \in \{t_1, s_2, t_2, \ldots, s_r, t_r, s_{r+1}\}$ of a representative of λ with the boundary of the polygon $P^D_{g,n}$ lies on a side a_j, b_j, respectively, if the last factor in $P(p)$ is $\bar{a}_j, \bar{b}_j$ and on a'_j, b'_j if it is $\bar{a}_j^{-1}, \bar{b}_j^{-1}$. Similarly, an intersection point with a side m_i lies on v_i and v'_i, respectively, if it is $\bar{m}_i$ and $\bar{m}_i^{-1}$

$$\begin{aligned} &p \in a_j \Leftrightarrow LF(P(p)) = \bar{a}_j \quad && p \in b_j \Leftrightarrow LF(P(p)) = \bar{b}_j \quad && p \in v_i \Leftrightarrow LF(P(p)) = \bar{m}_i \\ &p \in a'_j \Leftrightarrow LF(P(p)) = \bar{a}_j^{-1} \quad && p \in b'_j \Leftrightarrow LF(P(p)) = \bar{b}_j^{-1} \quad && p \in v'_i \Leftrightarrow LF(P(p)) = \bar{m}_i^{-1}. \end{aligned} \tag{19}$$

Furthermore, we note that if $p = t_k$ is the endpoint of the segment l_k, the starting point of the segment l_k lies on v_i, a_j, b_j, respectively, if $\bar{x}_{k-1}^{-\alpha_{k-1}} = \mathrm{LF}(P(p)^{-1})$ is $\bar{m}_i, \bar{a}_j, \bar{b}_j$ and on v'_i, a'_j, b'_j if it is $\bar{m}_i^{-1}, \bar{a}_j^{-1}, \bar{b}_j^{-1}$. Similarly, if $p = s_{k+1}$ is the starting point of the segment l_{k+1}, its endpoint t_{k+1} lies on v_i, a_j, b_j if $\bar{x}_{k+1}^{\alpha_{k+1}} = \mathrm{LF}(P(p)^{-1}) = \bar{m}_i, \bar{a}_j, \bar{b}_j$ and on v'_i, a'_j, b'_j for $\bar{x}_{k+1}^{\alpha_{k+1}} = \mathrm{LF}(P(p)^{-1}) = \bar{m}_i^{-1}, \bar{a}_j^{-1}, \bar{b}_j^{-1}$. Hence, the last factor $\mathrm{LF}(P(p)^{-1})$ of $P(p)^{-1}$ determines the location of the other end of the segment which contains p.

To pursue this reasoning further, we consider the elements $\lambda_p^{(s)} \in \mathrm{CPerm}(\lambda)$ obtained as cyclical permutations of $\lambda_p^{(1)} = P(p)$:

$$\lambda_p^{(s)} = \begin{cases} \lambda_{k-s} & p = t_k \\ \lambda_{k+s}^{-1} & p = s_{k+1} \end{cases} = \begin{cases} \bar{x}_{k-s}^{\alpha_{k-s}} \cdots \bar{x}_1^{\alpha_1} \bar{x}_r^{\alpha_r} \cdots \bar{x}_{k-s+1}^{\alpha_{k-s+1}} & p = t_k \\ \bar{x}_{k+s}^{-\alpha_{k+s}} \cdots \bar{x}_r^{-\alpha_r} \bar{x}_1^{-\alpha_1} \cdots \bar{x}_{k+s-1}^{-\alpha_{k+s-1}} & p = s_{k+1} \end{cases} \qquad k, s = 1, \ldots, r \tag{20}$$

where we identify $r + s = s, -s = r - s$. By inspecting the associated segments on the polygon $P^D_{g,n}$, we find that if $p = t_k$ is the endpoint of the segment l_k, the last factors in $\lambda_p^{(s)}$ and $\left(\lambda_p^{(s)}\right)^{-1}$, respectively, determine on which side of the polygon $P^D_{g,n}$ the endpoint t_{k-s+1} of the segment l_{k-s+1} and the starting point s_{k+s-1} of the segment l_{k+s-1} are located. Similarly, for starting points $p = s_{k+1}$, the last factors in $\lambda_p^{(s)}$ and $\left(\lambda_p^{(s)}\right)^{-1}$ give the location of, respectively, the starting point s_{k+s} and the endpoint t_{k+s}.

We now consider the intersection points of a representative of λ with the sides of the polygon $P^D_{g,n}$ with the ordering obtained by traversing the boundary $\partial P^D_{g,n}$ counterclockwise starting at x_0. Note that this ordering is unique, since embedded curves are represented by non-intersecting segments on $P^D_{g,n}$ and exchanging two intersection points would give rise to an intersection of the associated segments. For two intersection points $p, q \in \{t_1, s_2, t_2, \ldots, s_r, t_r, s_{r+1}\}$ which are located on different sides of polygon or on different parts v_i, v'_i of a side m_i, it follows from figure 4 that p occurs before q if and only if the side containing p occurs before side containing q. Since these sides are given by last factors in the associated cyclic permutations $P(p) = \lambda_p^{(1)}$, $P(q) = \lambda_q^{(1)}$, this is the case if and only if

$$\mathrm{LF}\big(\lambda_p^{(1)}\big) <_D \mathrm{LF}\big(\lambda_q^{(1)}\big) \tag{21}$$

with respect to the ordering

$$\begin{aligned} \bar{b}_g^{-1} >_D \bar{a}_g^{-1} >_D \bar{b}_g >_D \bar{a}_g >_D \bar{b}_{g-1}^{-1} >_D \cdots >_D \bar{b}_1^{-1} >_D \bar{a}_1^{-1} >_D \\ \bar{b}_1 >_D \bar{a}_1 >_D \bar{m}_n^{-1} >_D \bar{m}_n >_D \cdots >_D \bar{m}_1^{-1} >_D \bar{m}_1. \end{aligned} \tag{22}$$

We now consider the case where both p and q are either located on the same side $x \in \{a_1, a_1', \dots, b_g, b_g'\}$ of the polygon $P^D_{g,n}$ or, in the case of the punctures, on a single segment $x \in \{v_1, v_1', \dots, v_n, v_n'\}$. This is the case if and only if the last factors in the associated permutations agree $\mathrm{LF}(\lambda_p^{(1)}) = LF(\lambda_q^{(1)})$. The fact that the associated segments cannot intersect then implies that the order of p and q depends on the location of the other end of these segments, which we will denote by p', q' and which are given by $\mathrm{LF}((\lambda_p^{(1)})^{-1})$, $\mathrm{LF}((\lambda_q^{(1)})^{-1})$. Note that the fact that expression (15) for λ is cyclically reduced implies that p' and q' cannot lie on the side x:

$$\mathrm{LF}(\lambda_p^{(s)}) \neq \mathrm{LF}((\lambda_p^{(s)})^{-1}) \qquad \forall p \in \{t_1, s_2, t_2, \dots, s_r, t_r, s_{r+1}\}, \quad s = 1, \dots, r. \tag{23}$$

If we have $\mathrm{LF}((\lambda_p^{(1)})^{-1}) \neq \mathrm{LF}((\lambda_q^{(1)})^{-1})$, then p' and q' are located either on different sides of $P^D_{g,n}$ or on different parts v_i, v_i' associated with a side m_i. By drawing the associated segments on the polygon $P^D_{g,n}$, we then find that p occurs before q if either p' is located on a side before side x and q' on side after x or if both the sides containing p' and q' occur before or after x with the one containing q' before the one for p'. This can be implemented by defining an ordering $>_x$ of the set of sides $\{v_1, v_1', \dots, b_g, b_g'\} - \{x\}$ obtained by removing the factor associated with x from the ordering (22) and performing a cyclic permutation which moves the factors to the right of the factor associated with x in (22) to the left. Explicitly, we have for the factors $\overline{m}_i^{\pm 1}, \overline{a}_j^{\pm 1}, \overline{b}_j^{\pm 1}$:

$$\begin{aligned}
&\overline{m}_{i-1}^{-1} >_{m_i} \overline{m}_{i-1} >_{m_i} \cdots >_{m_i} \overline{m}_1 >_{m_i} \overline{b}_g^{-1} >_{m_i} \overline{a}_g^{-1} >_{m_i} \overline{b}_g >_{m_i} \overline{a}_g >_{m_i} \cdots >_{m_i} \overline{m}_{i+1}^{-1} >_{m_i} \overline{m}_{i+1} >_{m_i} \overline{m}_i^{-1}\\
&\overline{m}_i >_{m_i^{-1}} \overline{m}_{i-1}^{-1} >_{m_i^{-1}} \overline{m}_{i-1} >_{m_i^{-1}} \cdots >_{m_i^{-1}} \overline{m}_1 >_{m_i^{-1}} \overline{b}_g^{-1} >_{m_i^{-1}} \overline{a}_g^{-1} >_{m_i^{-1}} \overline{b}_g >_{m_i^{-1}} \overline{a}_g >_{m_i^{-1}} \cdots >_{m_i^{-1}} \overline{m}_{i+1}\\
&\overline{b}_{j-1}^{-1} >_{a_j} \overline{a}_{j-1}^{-1} >_{a_j} \overline{b}_{j-1} >_{a_j} \overline{a}_{j-1} >_{a_j} \cdots >_{a_j} \overline{m}_1 >_{a_j} \overline{b}_g^{-1} >_{a_j} \cdots >_{a_j} \overline{a}_{j+1} >_{a_j} \overline{b}_j^{-1} >_{a_j} \overline{a}_j^{-1} >_{a_j} \overline{b}_j\\
&\overline{b}_j >_{a_j^{-1}} \overline{a}_j >_{a_j^{-1}} \overline{b}_{j-1}^{-1} >_{a_j^{-1}} \overline{a}_{j-1}^{-1} >_{a_j^{-1}} \overline{b}_{j-1} >_{a_j^{-1}} \overline{a}_{j-1} >_{a_j^{-1}} \cdots >_{a_j^{-1}} \overline{m}_1 >_{a_j^{-1}} \overline{b}_g^{-1} >_{a_j^{-1}} \cdots >_{a_j^{-1}} \overline{a}_{j+1} >_{a_j^{-1}} \overline{b}_j^{-1}\\
&\overline{a}_j >_{b_j} \overline{b}_{j-1}^{-1} >_{b_j} \overline{a}_{j-1}^{-1} >_{b_j} \overline{b}_{j-1} >_{b_j} \overline{a}_{j-1} >_{b_j} \cdots >_{b_j} \overline{m}_1 >_{b_j} \overline{b}_g^{-1} >_{b_j} \cdots >_{b_j} \overline{a}_{j+1} >_{b_j} \overline{b}_j^{-1} >_{b_j} \overline{a}_j^{-1}\\
&\overline{a}_j^{-1} >_{b_j^{-1}} \overline{b}_j >_{b_j^{-1}} \overline{a}_j >_{b_j^{-1}} \overline{b}_{j-1}^{-1} >_{b_j^{-1}} \overline{a}_{j-1}^{-1} >_{b_j^{-1}} \overline{b}_{j-1} >_{b_j^{-1}} \overline{a}_{j-1} >_{b_j^{-1}} \cdots >_{b_j^{-1}} \overline{m}_1 >_{b_j^{-1}} \overline{b}_g^{-1} >_{b_j^{-1}} \cdots >_{b_j^{-1}} \overline{a}_{j+1}.
\end{aligned} \tag{24}$$

The intersection point p then occurs before q on x if and only if the last factor $\mathrm{LF}((\lambda_p^{(1)})^{-1})$ is greater than the last factor $\mathrm{LF}((\lambda_q^{(1)})^{-1})$ with respect to the ordering associated with the side x:

$$\mathrm{LF}((\lambda_p^{(1)})^{-1}) >_x \mathrm{LF}((\lambda_q^{(1)})^{-1}). \tag{25}$$

Finally, we consider the case where both p, q are located on a side x of $P^D_{g,n}$ and the other ends of the associated segments both lie on another side $y \in \{v_1, v_1', \dots, b_g, b_g'\} - \{x\}$. This is the case if and only if $\mathrm{LF}((\lambda_p^{(1)})^{-1}) = \mathrm{LF}((\lambda_q^{(1)})^{-1})$ or, equivalently, $\mathrm{LF}(\lambda_p^{(2)}) = \mathrm{LF}(\lambda_q^{(2)})$. Then the fact that segments cannot intersect implies that p occurs before q on x if and only if q' occurs before p' on y. We then consider the corresponding points p'', q'' on the side y' of the polygon $P^D_{g,n}$ identified with y. As the orientation of two sides y, y' in figure 4 is the opposite, we find that q' occurs before p' on y if and only if p'' occurs before q'' on y'. Hence, we can repeat the reasoning of the last paragraphs for the points p'' and q''. In the case where the original intersection point p is an endpoint $p = t_k$, the corresponding point on y' is the endpoint $p'' = t_{k-1}$ of the previous segment, and for a starting point $p = s_{k+1}$ it is the starting point of the next segment $p'' = s_{k+2}$. Hence, the side containing p'', q'' is given by the last factors in $\lambda_p^{(2)}, \lambda_q^{(2)}$ and other ends of the associated segments by last factors in $(\lambda_p^{(2)})^{-1}$,

$\left(\lambda_q^{(2)}\right)^{-1}$. If these ends are located on different sides, we have $\mathrm{LF}\left(\left(\lambda_p^{(2)}\right)^{-1}\right) \neq \mathrm{LF}\left(\left(\lambda_q^{(2)}\right)^{-1}\right)$ and p occurs before q if and only if

$$\mathrm{LF}\left(\left(\lambda_p^{(2)}\right)^{-1}\right) >_{\mathrm{LF}(\lambda_p^{(2)})} \mathrm{LF}\left(\left(\lambda_q^{(2)}\right)^{-1}\right). \tag{26}$$

Otherwise we apply the same reasoning to the intersection points associated with $\lambda_p^{(3)}, \lambda_q^{(3)}$. By repeatedly applying this argument to the permutations $\lambda_p^{(s)}, \lambda_q^{(s)}$ with increasing s, we either obtain permutations $\lambda_p^{(k)}, \lambda_q^{(k)}$ whose last factors differ or the expression (15) of λ as a cyclically reduced word in the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$ is periodic $\lambda = q^k$, $1 < k < r$, with a cyclically reduced word $q = \overline{y}_s^{\beta_s} \cdots \overline{y}_1^{\beta_1}$ in $\overline{m}_i, \overline{a}_j, \overline{b}_j$. However, it is easy to see that periodic words cannot have embedded representatives. The representation of $\lambda = q^k$ by segments on the polygon $P_{g,n}^D$ is obtained from the representation of q by drawing each of the segments associated with pairs of factors in qk times and omitting the segment associated with $\overline{y}_1^{\beta_1}$ and $\overline{y}_s^{\beta_s}$. One then has to add k segments each connecting one of the k points associated with factor $\overline{y}_1^{\beta_1}$ with one of the points associated with $\overline{y}_s^{\beta_s}$. However, it is clear that it is impossible to do this without creating intersections of the segments.

Hence, the algorithm described above terminates and there exists a $k \in \{1, \ldots, r-1\}$ such that $\mathrm{LF}\left(\lambda_p^{(s)}\right) = \mathrm{LF}\left(\lambda_q^{(s)}\right)$ for $s \leqslant k$ and $\mathrm{LF}\left(\left(\lambda_p^{(k)}\right)^{-1}\right) \neq \mathrm{LF}\left(\left(\lambda_q^{(k)}\right)^{-1}\right)$. The intersection point p then occurs before q if and only if the last factor $\mathrm{LF}\left(\left(\lambda_p^{(k)}\right)^{-1}\right)$ is greater than the last factor $\mathrm{LF}\left(\left(\lambda_q^{(k)}\right)^{-1}\right)$ with respect to the ordering $>_{\mathrm{LF}(\lambda_p^{(k)})}$, and we obtain the following theorem.

Theorem 3.1. *Consider an element $\lambda \in \pi_1(S_{g,n} \setminus D)$ with an embedded representative and two intersection points $p, q \in \{t_1, s_2, t_2, \ldots, s_r, t_r, s_{r+1}\}$ of this representative with the polygon $P_{g,n}^D$. Denote by $\lambda_p^{(s)}, \lambda_q^{(s)}$ the cyclic permutations of λ and its inverse assigned to the points p and q as defined in (20). Then, the intersection point p occurs before q with respect to the ordering obtained by traversing the polygon $P_{g,n}^D$ counterclockwise starting at x_0 if and only if either*

$$\mathrm{LF}\left(\lambda_p^{(1)}\right) <_D \mathrm{LF}\left(\lambda_q^{(1)}\right), \tag{27}$$

in which case the points p, q are located on different sides of the polygon $P_{g,n}^D$ or on different parts v_i, v_i' of a side m_i, or

$$\exists k \in \{1, \ldots, r-1\}: \quad \mathrm{LF}\left(\lambda_p^{(s)}\right) = \mathrm{LF}\left(\lambda_q^{(s)}\right) \forall s \leqslant k, \quad \mathrm{LF}\left(\left(\lambda_p^{(k)}\right)^{-1}\right) >_{\mathrm{LF}(\lambda_p^{(k)})} \mathrm{LF}\left(\left(\lambda_q^{(k)}\right)^{-1}\right). \tag{28}$$

Hence, for any element $\lambda \in \pi_1(S_{g,n} \setminus D)$ with an embedded representative, the unique order in which the starting and endpoints of the associated segments occur on the polygon $P_{g,n}^D$ gives rise to an ordering of the set of cyclic permutations $\mathrm{CPerm}(\lambda)$ defined by the conditions (27), (28). In particular, this induces an ordering of the intersection points of λ with each of the generators $m_i, a_j, b_j \in \pi_1(S_{g,n} \setminus D)$. Intersection points of λ with the generators a_j, b_j are in one-to-one correspondence with intersection points of λ with the sides a_j, b_j of $P_{g,n}^D$ and therefore with elements $\tau \in \mathrm{CPerm}(\lambda)$ satisfying, respectively, $\mathrm{LF}(\tau) = \overline{a}_j$ and $\mathrm{LF}(\tau) = \overline{b}_j$. Taking into account the orientation of the sides a_j, b_j on $P_{g,n}^D$ we find that the order in which these intersection points occur on a_j agrees with the one on the polygon, while for b_j it is the opposite. Similarly, intersection points of λ with the generator m_i correspond one to one to intersection points of λ with the segments v_i and v_i' and hence to elements $\tau \in \mathrm{CPerm}(\lambda)$, $\mathrm{LF}(\tau) = \overline{m}_i^{\pm 1}$, and the order in which these intersection points occur on the generator m_i is the order of the corresponding points on $P_{g,n}^D$. We therefore obtain a purely algebraic procedure, which allows one to determine the number and order of intersection points of general embedded curves on $S_{g,n}\backslash D$ with the representatives of the generators m_i, a_j, b_j and the associated oriented intersection numbers from the expression of its homotopy equivalence class as a reduced word in the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$.

3.2. The surface picture: assigning the intersection points between different factors

The involution $I \in \mathrm{Aut}(\pi_1(S_{g,n}\backslash D))$ not only allows us to determine the number and order of intersection points of elements $\lambda \in \pi_1(S_{g,n}\backslash D)$ with embedded representatives with the generators m_i, a_j, b_j but also assigns this intersection points (almost) uniquely between the different factors in the expression of λ as a reduced word in the original m_i, a_j, b_j. Furthermore, we will show that this assignment of intersection points can be implemented by graphically decomposing a representative of λ into curves representing m_i, a_j, b_j.

The idea is the following. We consider the expression (15) of λ as a reduced word in the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$ and the intersection point or pair of intersection points associated with a factor $\overline{x}_k^{\alpha_k}$ in (15). We then split the expression of this factor as a reduced word in m_i, a_j, b_j into two reduced words $\overline{x}_k^{\alpha_k} = w_2 w_1$, which correspond to the two segments on the polygon $P_{g,n}^D$ in the decompositions (10), (11), (12). If y_2 and y_1, respectively, denote the expressions of the elements $w_1\overline{x}_{k-1}^{\alpha_{k-1}}\cdots\overline{x}_1^{\alpha_1} \in \pi_1(S_{g,n}\backslash D)$ and $\overline{x}_r^{\alpha_r}\cdots\overline{x}_{k+1}^{\alpha_{k+1}}w_2 \in \pi_1(S_{g,n}\backslash D)$ in (15) as reduced words in m_i, a_j, b_j, the product $\lambda = y_2 y_1$ then gives an expression of λ as a reduced word in the generators m_i, a_j, b_j and we assign the intersection point or pair of intersection points corresponding to $\overline{x}_k^{\alpha_k}$ between the reduced words y_2 and y_1 in this expression.

However, it is *a priori* not guaranteed that the product of the reduced words y_2, y_1 in m_i, a_j, b_j agrees with the expression of λ as a reduced word in m_i, a_j, b_j, since it is possible that this product is not reduced. This is the case if and only if the reduced words y_2, y_1 are of the form $y_2 = z_2 x^{\epsilon}$, $y_1 = x^{-\epsilon} z_1$, where $x \in \{m_1, \ldots, b_g\}$, $\epsilon \in \{\pm 1\}$ and z_1, z_2 are reduced words in m_i, a_j, b_j. In order to obtain a well-defined assignment of the intersection points between the different factors in the expression of λ as a reduced word in m_i, a_j, b_j, we therefore have to show that this situation can be avoided. Furthermore, since there are two ways of splitting the factors $\overline{a}_j, \overline{b}_j$ in (11), (12), the question arises if the requirement that the resulting expression for λ is a reduced word in m_i, a_j, b_j removes this ambiguity in the splitting. It turns out that up to a small residual ambiguity this is the case. We obtain an almost unique assignment of intersection points between the different factors m_i, a_j, b_j in λ which is summarized in the following theorem.

Theorem 3.2. *Consider an embedded element $\lambda \in \pi_1(S_{g,n}\backslash D)$ and a factor $\overline{x}_k^{\alpha_k}$ in its expression (15) as a reduced word in the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$.*

(1) If $\overline{x}_k = \overline{m}_i$, split the factor according to

$$\overline{m}_i = \left(m_1^{-1}\cdots m_i^{-1}\right)(m_{i-1}\cdots m_1) \tag{29}$$

and let $y_s^{\beta_s}\cdots y_{l+1}^{\beta_{l+1}}$ and $y_l^{\beta_l}\cdots y_1^{\beta-1}$, $y_k \in \{m_1, \ldots, b_g\}$, $\beta_k \in \{\pm 1\}$ be the reduced words in m_i, a_j, b_j obtained by setting

$$y_l^{\beta_l}\cdots y_1^{\beta_1} = \begin{cases} m_{i-1}\cdots m_1\overline{x}_{k-1}^{\alpha_{k-1}}\cdots\overline{x}_1^{\alpha_1} & \textit{if} \quad \alpha_k = 1 \\ m_i\cdots m_1\overline{x}_{k-1}^{\alpha_{k-1}}\cdots\overline{x}_1^{\alpha_1} & \textit{if} \quad \alpha_k = -1 \end{cases} \tag{30}$$

$$y_s^{\beta_s}\cdots y_{l+1}^{\beta_{l+1}} = \begin{cases} \overline{x}_r^{\alpha_r}\cdots\overline{x}_{k+1}^{\alpha_{k+1}}m_1^{-1}\cdots m_i^{-1} & \textit{if} \quad \alpha_k = 1 \\ \overline{x}_r^{\alpha_r}\cdots\overline{x}_{k+1}^{\alpha_{k+1}}m_1^{-1}\cdots m_{i-1}^{-1} & \textit{if} \quad \alpha_k = -1. \end{cases} \tag{31}$$

Then, the expression for λ *as a reduced word in the generators* m_i, a_j, b_j *is given by the product* $\lambda = y_s^{\beta_s} \cdots y_{l+1}^{\beta_{l+1}} y_l^{\beta_l} \cdots y_1^{\beta_1}$ *and we assign the two corresponding intersection points of* λ *with* m_i *between the factors* $y_{l+1}^{\beta_{l+1}}$ *and* $y_l^{\beta_l}$ *and to the starting and endpoint of* m_i.

(2) *If* $\overline{x}_k \in \{\overline{a}_j, \overline{b}_j\}$, *let* $y_l^{\beta_l} \cdots y_1^{\beta_1}$ *and* $y_s^{\beta_s} \cdots y_{l+1}^{\beta_{l+1}}$ *denote the reduced words in* m_i, a_j, b_j *obtained by splitting* $\overline{x}_k^{\alpha_k}$ *according to*

$$\begin{aligned} \overline{a}_j &= \left(m_1^{-1} \cdots h_{j-1}^{-1} a_j^{-1} b_j a_j\right)(h_{j-1} \cdots m_1) \\ \overline{b}_j &= \left(m_1^{-1} \cdots h_{j-1}^{-1} a_j^{-1} b_j a_j\right)\left(b_j^{-1} a_j h_{j-1} \cdots m_1\right) \end{aligned} \tag{32}$$

$$y_l^{\beta_l} \cdots y_1^{\beta_1} = \begin{cases} h_{j-1} \cdots m_1 \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1} & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{a}_j \\ a_j^{-1} b_j^{-1} a_j h_{j-1} \cdots m_1 \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1} & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{a}_j^{-1} \\ b_j^{-1} a_j h_{j-1} \cdots m_1 \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1} & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{b}_j \\ a_j^{-1} b_j^{-1} a_j h_{j-1} \cdots m_1 \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1} & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{b}_j^{-1} \end{cases} \tag{33}$$

$$y_s^{\beta_s} \cdots y_{l+1}^{\beta_{l+1}} = \begin{cases} \overline{x}_r^{\alpha_r} \cdots \overline{x}_{k+1}^{\alpha_{k+1}} m_1^{-1} \cdots h_{j-1}^{-1} a_j^{-1} b_j a_j & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{a}_j \\ \overline{x}_r^{\alpha_r} \cdots \overline{x}_{k+1}^{\alpha_{k+1}} m_1^{-1} \cdots h_{j-1}^{-1} & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{a}_j^{-1} \\ \overline{x}_r^{\alpha_r} \cdots \overline{x}_{k+1}^{\alpha_{k+1}} m_1^{-1} \cdots h_{j-1}^{-1} a_j^{-1} b_j a_j & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{b}_j \\ \overline{x}_r^{\alpha_r} \cdots \overline{x}_{k+1}^{\alpha_{k+1}} m_1^{-1} \cdots h_{j-1}^{-1} a_j^{-1} b_j & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{b}_j^{-1}, \end{cases} \tag{34}$$

and let $z_m^{\delta_m} \cdots z_1^{\delta_1}$ *and* $z_t^{\delta_t} \cdots z_{m+1}^{\beta_{m+1}}$ *be the reduced words obtained by splitting* $\overline{x}_k^{\alpha_k}$ *as*

$$\begin{aligned} \overline{a}_j &= \left(m_1^{-1} \cdots h_{j-1}^{-1} a_j^{-1} b_j\right)(a_j h_{j-1} \cdots m_1) \\ \overline{b}_j &= \left(m_1^{-1} \cdots h_{j-1}^{-1} a_j^{-1} b_j a_j b_j^{-1}\right)(a_j h_{j-1} \cdots m_1) \end{aligned} \tag{35}$$

$$z_m^{\delta_m} \cdots z_1^{\delta_1} = \begin{cases} a_j h_{j-1} \cdots m_1 \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1} & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{a}_j \\ b_j^{-1} a_j h_{j-1} \cdots m_1 \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1} & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{a}_j^{-1} \\ a_j h_{j-1} \cdots m_1 \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1} & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{b}_j \\ h_j h_{j-1} \cdots m_1 \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1} & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{b}_j^{-1} \end{cases} \tag{36}$$

$$z_t^{\delta_t} \cdots z_{m+1}^{\beta_{m+1}} = \begin{cases} \overline{x}_r^{\alpha_r} \cdots \overline{x}_{k+1}^{\alpha_{k+1}} m_1^{-1} \cdots h_{j-1}^{-1} a_j^{-1} b_j & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{a}_j \\ \overline{x}_r^{\alpha_r} \cdots \overline{x}_{k+1}^{\alpha_{k+1}} m_1^{-1} \cdots h_{j-1}^{-1} a_j^{-1} & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{a}_j^{-1} \\ \overline{x}_r^{\alpha_r} \cdots \overline{x}_{k+1}^{\alpha_{k+1}} m_1^{-1} \cdots h_{j-1}^{-1} h_j^{-1} & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{b}_j \\ \overline{x}_r^{\alpha_r} \cdots \overline{x}_{k+1}^{\alpha_{k+1}} m_1^{-1} \cdots h_{j-1}^{-1} a_j^{-1} & \text{if} \quad \overline{x}_k^{\alpha_k} = \overline{b}_j^{-1}. \end{cases} \tag{37}$$

Then the expression of λ *as a reduced word in* m_i, a_j, b_j *is either given by* $\lambda = y_s^{\beta_s} \cdots y_{l+1}^{\beta_{l+1}} y_l^{\beta_l} \cdots y_1^{\beta_1}$ *and we assign the corresponding intersection point between the factors* $y_{l+1}^{\beta_{l+1}}$ *and* $y_l^{\beta_l}$ *and to the starting point of* a_j *or* b_j *or it is given by* $\lambda = z_t^{\delta_t} \cdots z_{m+1}^{\beta_{m+1}} z_m^{\delta_m} \cdots z_1^{\delta_1}$ *and we assign the corresponding intersection point between the factors* $z_{m+1}^{\beta_{m+1}}$ *and* $z_m^{\delta_m}$ *and to the endpoint of* a_j *or* b_j. *Ambiguity in the sense that both* $y_s^{\beta_s} \cdots y_1^{\beta_1}$ *and* $z_t^{\delta_t} \cdots z_1^{\delta_1}$ *are reduced words in* m_i, a_j, b_j *arises if and only if either* $y_{l+1}^{\beta_{l+1}} = z_m^{\delta_m} = x_k$ *or* $y_l^{\beta_l} = z_{m+1}^{\delta_{m+1}} = x_k^{-1}$.

The proof of theorem 3.2 is rather lengthy and technical and makes use of the following lemma.

Lemma 3.3. *Consider an element of $\pi_1(S_{g,n}\backslash D)$ given as a reduced word in the generators m_i, a_j, b_j and their duals by*

$$\overline{x}_r^{\alpha_r}\cdots\overline{x}_1^{\alpha_1} = wzw' \qquad \overline{x}_k \in \{\overline{m}_1,\ldots,\overline{b}_g\}, \quad \alpha_k \in \{\pm 1\}, \quad z \in \left\{m_1^{\pm 1},\ldots,b_g^{\pm 1}\right\}, \tag{38}$$

where w, w' are reduced words in m_i, a_j, b_j. Then, we have the following implications:

$$w' = m_i\cdots m_1, \overline{x}_1^{\alpha_1} \neq \overline{m}_i^{-1} \quad\Rightarrow\quad z \in \left\{m_{i+1}^{\pm 1},\ldots,b_g^{\pm 1}\right\} \tag{39}$$

$$w' = m_{i-1}\cdots m_1, \overline{x}_1^{\alpha_1} \neq \overline{m}_i \quad\Rightarrow\quad z \in \left\{m_1^{\pm 1},\ldots,m_{i-1}^{\pm 1}, m_i\right\} \tag{40}$$

$$w' = h_{j-1}\cdots m_1, \overline{x}_1^{\alpha_1} \neq \overline{a}_j \quad\Rightarrow\quad z \in \left\{m_1^{\pm 1},\ldots,b_{j-1}^{\pm 1}, a_j\right\} \tag{41}$$

$$w' = a_j h_{j-1}\cdots m_1, \overline{x}_1^{\alpha_1} \neq \overline{a}_j \quad\Rightarrow\quad z \in \left\{b_j^{-1}, a_{j+1}^{\pm 1},\ldots,b_g^{\pm 1}\right\} \tag{42}$$

$$w' = a_j^{-1}b_j^{-1}a_j h_{j-1}\cdots m_1, \overline{x}_1^{\alpha_1} \neq \overline{a}_j^{-1} \quad\Rightarrow\quad z \in \left\{a_j^{-1}, b_j^{\pm 1}, a_{j+1}^{\pm 1},\ldots,b_g^{\pm 1}\right\} \tag{43}$$

$$w' = b_j^{-1}a_j h_{j-1}\cdots m_1, \overline{x}_1^{\alpha_1} \neq \overline{a}_j^{-1} \quad\Rightarrow\quad z \in \left\{m_1^{\pm 1},\ldots,b_{j-1}^{\pm 1}, a_j^{\pm 1}\right\} \tag{44}$$

$$w' = b_j^{-1}a_j h_{j-1}\cdots m_1, \overline{x}_1^{\alpha_1} \neq \overline{b}_j \quad\Rightarrow\quad z \in \left\{a_j^{-1}, b_j^{-1}, a_{j+1}^{\pm 1},\ldots,b_g^{\pm 1}\right\} \tag{45}$$

$$w' = a_j h_{j-1}\cdots m_1, \overline{x}_1^{\alpha_1} \neq \overline{b}_j \quad\Rightarrow\quad z \in \left\{m_1^{\pm 1},\ldots,b_{j-1}^{\pm 1}, a_j, b_j^{\pm 1}\right\} \tag{46}$$

$$w' = a_j^{-1}b_j^{-1}a_j h_{j-1}\cdots m_1, \overline{x}_1^{\alpha_1} \neq \overline{b}_j^{-1} \quad\Rightarrow\quad z \in \left\{m_1^{\pm 1},\ldots,b_{j-1}^{\pm 1}, b_j\right\} \tag{47}$$

$$w' = h_j\cdots m_1, \overline{x}_1^{\alpha_1} \neq \overline{b}_j^{-1} \quad\Rightarrow\quad z \in \left\{a_{j+1}^{\pm 1},\ldots,b_g^{\pm 1}\right\}. \tag{48}$$

Proof. The proof is by induction over the length r of λ as a reduced word in $\overline{m}_i, \overline{a}_j, \overline{b}_j$ and similar for all of the implications above. We prove the statement (41). For $r = 1$, expressions (8) for $\overline{m}_i, \overline{a}_j, \overline{b}_j$ imply $z \in \left\{a_j, a_{j-1}^{-1}\right\}$, since the word wzw' is reduced. Now assume the statement is true for $r \leqslant k$ and there exists an element of $\pi_1(S_{g,n}\backslash D)$ whose expression as a reduced word in m_i, a_j, b_j and their duals are given by $\overline{x}_{k+1}^{\alpha_{k+1}}\cdots\overline{x}_1^{\alpha_1} = wzh_{j-1}\cdots h_1$, where w is a reduced word in m_i, a_j, b_j, $\overline{x}_1^{\alpha_1} \neq \overline{a}_j$ and $z \in \left\{a_j^{-1}, b_j^{\pm 1}, a_{j+1}^{\pm 1}\cdots b_g^{\pm 1}\right\}$. This implies that the elements $\overline{x}_k^{\alpha_k}\cdots\overline{x}_1^{\alpha_1}$ and $\overline{x}_{k+1}^{\alpha_{k+1}}$, expressed as reduced words in m_i, a_j, b_j, are of the form $\overline{x}_k^{\alpha_k}\cdots\overline{x}_1^{\alpha_1} = yh_{j-1}\cdots h_1$, $\overline{x}_{k+1}^{\alpha_{k+1}} = y'zy^{-1}$, where y, y' are reduced words in m_i, a_j, b_j. If $z \in \left\{a_{j+1}^{\pm 1}\cdots b_g^{\pm 1}\right\}$, it follows from the expression (8) that the last letter in y is an element of $\left\{b_j^{-1}, a_{j+1}^{\pm 1},\ldots,b_g^{\pm 1}\right\}$, and we obtain a contradiction. Similarly, for $z = a_j^{-1}$ the last letter of y is in $\left\{b_j^{\pm 1}\right\}$, for $z = b_j^{-1}$ in $\left\{a_{j+1}, a_j^{-1}\right\}$ which again contradicts the induction hypothesis. Finally, for $z = b_j$, the last letter in y is either again in $\left\{a_j^{-1}, a_{j+1}\right\}$ or we have $\overline{x}_{k+1}^{\alpha_{k+1}} = \overline{b}_j^{-1}$ $y = a_j^{-1}b_j a_j h_{j-1}\cdots m_1$, which implies $\overline{x}_k^{\alpha_k}\cdots\overline{x}_1^{\alpha_1} = \overline{a}_j$ and contradicts the induction hypothesis. Hence, the statement is true for $r = k+1$, which proves the claim. □

Proof of theorem 3.2. We prove the statement for the case $\overline{x}_k^{\alpha_k} = \overline{a}_j$. The reasoning for the other cases is analogous. Suppose the word $y_s^{\beta_s}\cdots y_{l+1}^{\beta_{l+1}}y_l^{\beta_l}\cdots y_1^{\beta_1}$ in m_i, a_j, b_j defined by (33), (34) is not reduced, i.e. $y_l^{\beta_l} = y_{l+1}^{-\beta_{l+1}}$. Then, either the expression for the product $\overline{x}_r^{\alpha_r}\cdots\overline{x}_{k+1}^{\alpha_{k+1}}$ as a reduced word in m_i, a_j, b_j must be of the form

$$\overline{x}_r^{\alpha_r}\cdots\overline{x}_{k+1}^{\alpha_{k+1}} = uxa_j^{-1}b_j^{-1}a_j h_{j-1}\cdots m_1, \tag{49}$$

or the expression for $\overline{x}_1^{-\alpha_1}\cdots\overline{x}_{k-1}^{-\alpha_{k-1}}$ as a reduced word in m_i, a_j, b_j must be of the form

$$\overline{x}_1^{-\alpha_1}\cdots\overline{x}_{k-1}^{-\alpha_{k-1}} = u'x'h_{j-1}\cdots m_1 \tag{50}$$

where $x, x' \in \{m_1,\ldots,b_g\}$ and u, u' are reduced words in m_i, a_j, b_j. Now note that the expression (15) for λ is reduced and therefore $\overline{x}_{k-1}^{\alpha_{k-1}}, \overline{x}_{k+1}^{\alpha_{k+1}} \neq \overline{a}_j^{-1}$, which allows us to apply the identities (41), (43) in lemma 3.3 to expressions (50), (50). Suppose the expression for $\overline{x}_r^{\alpha_r}\cdots\overline{x}_{k+1}^{\alpha_{k+1}}$ as a reduced word in m_i, a_j, b_j is of the form (49). Then, the identity (43) implies $x\in\left\{a_j^{-1}, b_{j+1}^{\pm 1},\ldots,b_g^{\pm 1}\right\}$ and therefore $y_s^{\beta_s}\cdots y_1^{\beta_1}$ is reduced unless the expression for $\overline{x}_1^{-\alpha_1}\cdots\overline{x}_{k-1}^{-\alpha_{k-1}}$ as a reduced word in m_i, a_j, b_j is of the form (50). But then identity (41) implies $x'\in\left\{m_1^{\pm 1},\ldots,b_{j-1}^{\pm 1}, a_j\right\}$ and therefore $y_s^{\beta_s}\cdots y_1^{\beta_1}$ is reduced.

Hence, the expression of $\overline{x}_1^{-\alpha_1}\cdots\overline{x}_{k-1}^{-\alpha_{k-1}}$ as a reduced word in the generators m_i, a_j, b_j must be of the form (50) with $x' = a_j$ and the corresponding expression for $\overline{x}_r^{\alpha_r}\cdots\overline{x}_{k+1}^{\alpha_{k+1}}$ must be given by

$$\overline{x}_r^{\alpha_r}\cdots\overline{x}_{k+1}^{\alpha_{k+1}} = u''x''b_j^{-1}a_jh_{j-1}\cdots m_1, \tag{51}$$

where u'' is a reduced word in m_i, a_j, b_j and $x''\in\left\{m_1^{\pm 1},\ldots,b_g^{\pm 1}\right\}\backslash\left\{a_j^{-1}\right\}$. This implies $y_l^{-\beta_l} = y_{l+1}^{\beta_{l+1}} = a_j$. Furthermore, by applying identities (42) and (44) in lemma 3.3 to the expressions of , respectively, $\overline{x}_1^{-\alpha_1}\cdots\overline{x}_{k-1}^{-\alpha_{k-1}}$ and $\overline{x}_r^{\alpha_r}\cdots\overline{x}_{k+1}^{\alpha_{k+1}}$ as reduced words in m_i, a_j, b_j, we find that they are of the form

$$\begin{aligned} &\overline{x}_1^{-\alpha_1}\cdots\overline{x}_{k-1}^{-\alpha_{k-1}} = u'''x'''a_jh_{j-1}\cdots m_1 && x'''\in\left\{b_j^{-1}, a_{j+1}^{\pm 1},\ldots,b_g^{\pm 1}\right\}\\ &\overline{x}_r^{\alpha_r}\cdots\overline{x}_{k+1}^{\alpha_{k+1}} = u''x''b_j^{-1}a_jh_{j-1}\cdots m_1 && x''\in\left\{m_1^{\pm 1},\ldots,b_{j-1}^{\pm 1}, a_j\right\}.\end{aligned} \tag{52}$$

This implies that the product of the reduced words

$$\begin{aligned} &y_{l-1}^{\beta_{l-1}}\cdots y_1^{\beta_1} = a_jh_{j-1}\cdots m_1\overline{x}_{k-1}^{\alpha_{k-1}}\cdots\overline{x}_1^{\alpha_1}\\ &y_r^{\beta_r}\cdots y_{l+2}^{\beta_{l+2}} = \overline{x}_r^{\alpha_r}\cdots\overline{x}_{k+1}^{\alpha_{k+1}}b_j^{-1}a_jh_{j-1}\cdots m_1\end{aligned} \tag{53}$$

is reduced and gives the expression of λ as a reduced word in m_i, a_j, b_j, which proves the claim. □

Hence, by splitting the dual generators as in (10)–(12), we obtain an almost unique assignment of the intersection points of a general embedded curve λ with the generators m_i, a_j, b_j between the different factors in the expression of λ as a reduced word in m_i, a_j, b_j and to the starting and endpoints of m_i, a_j, b_j. We will now show that this assignment of intersection points corresponds to a graphical decomposition of a curve on $S_{g,n}\backslash D$ representing λ into representatives of m_i, a_j, b_j.

For this, we represent the generators $m_i, a_j, b_j, \overline{m}_i, \overline{a}_j, \overline{b}_j$ by curves as in figure 1, but instead of a basepoint we draw a line on which the starting points $s_{m_i}, s_{a_j}, s_{b_j}$ and endpoints $t_{m_i}, t_{a_j}, t_{b_j}$ are ordered from right to left according to

$$s_{m_1} < t_{m_1} < \cdots < s_{m_n} < t_{m_n} < s_{a_1} < s_{b_1} < t_{a_1} < t_{b_1} < \cdots < s_{a_g} < s_{b_g} < t_{a_g} < t_{b_g}, \tag{54}$$

and the basepoint $p\in S_{g,n}\backslash D$ is located to the right of s_{m_1}. The curves representing the generators $x\in\{m_i, a_j, b_j\}$ are decomposed into an oriented horizontal segment from p to the starting point s_x, a curve which starts in s_x and ends in t_x and another horizontal segment from t_x back to p as shown in figure 7. For their inverses, we set $s_{x^{-1}} = t_x$, $t_{x^{-1}} = s_x$. To obtain an embedded representative of an element $\lambda\in\pi_1(S_{g,n}\backslash D)$ given uniquely as a reduced word in the generators m_i, a_j, b_j by

$$\lambda = y_s^{\beta_s}\cdots y_1^{\beta_1} \qquad y_k\in\{m_1,\ldots,b_g\},\quad \beta_k\in\{\pm 1\}, \tag{55}$$

we draw consecutively the representatives of the factors $y_k^{\beta_k}$ and contract the overlapping horizontal segments such that the resulting curve has no self-intersections, see figures 7 and 8. Thus, we obtain a curve representing λ which is composed of curves representing the factors $y_k^{\beta_k}$ which start and end above the corresponding starting points s_k and endpoints t_k on the

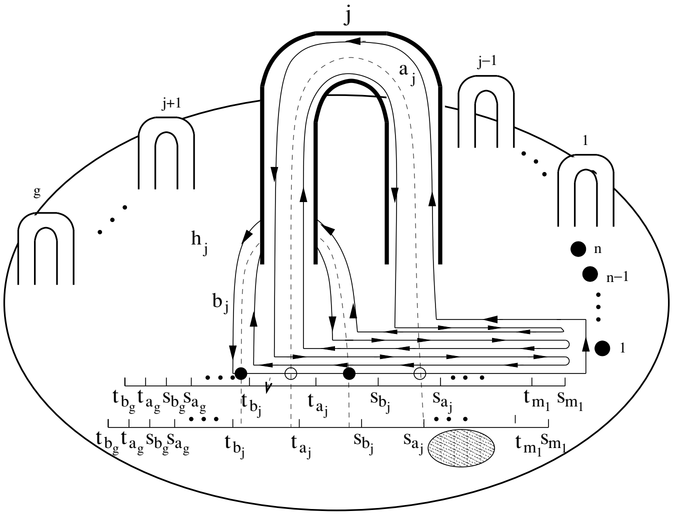

Figure 7. The representation of the element $\lambda = h_j = b_j \circ a_j^{-1} \circ b_j^{-1} \circ a_j$ and its intersection points with the generators a_j, b_j.

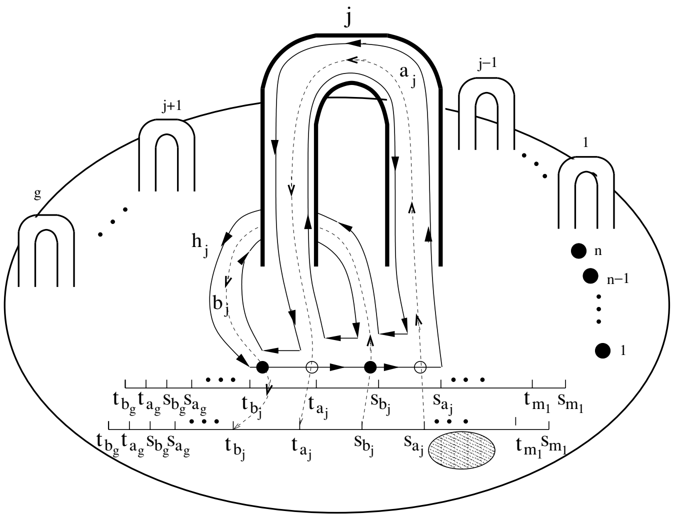

Figure 8. The representation of the element $\lambda = h_j = b_j \circ a_j^{-1} \circ b_j^{-1} \circ a_j$ and its nontrivial intersection points with the generators a_j, b_j.

horizontal line and horizontal segments $t_k s_{k+1}$ connecting the starting and endpoints of these factors.

To locate the intersection points of $\lambda \in \pi_1(S_{g,n} \setminus D)$ with the generators m_i, a_j, b_j between the different factors of λ and at the starting and endpoints of m_i, a_j, b_j, we draw two such lines, one for the generators m_i, a_j, b_j and one for λ such that the first one is tangent to the disc, while the one for λ is displaced slightly. We represent the generators m_i, a_j, b_j and λ graphically as described above such that all intersection points of λ with m_i, a_j, b_j lie on the horizontal segments $t_k s_{k+1}$ in the decomposition of λ and above the starting and endpoints $s_{m_i}, s_{a_j}, s_{b_j}, t_{m_i}, t_{a_j}, t_{b_j}$. We also require that for each factor $m_i^{\pm 1}$ which gives rise to a pair of intersection points with m_i, one of these points lies above the starting point s_{m_i} and one above the endpoint t_{m_i}. An intersection point q_i is then said to occur between the factors $y_i^{\beta_i}$ and $y_{i+1}^{\beta_{i+1}}$ on λ if it lies on the straight line connecting $t_i = t_{y_i^{\beta_i}}$ and $s_{i+1} = s_{y_{i+1}^{\beta_{i+1}}}$, where we set $t_0 = t_n$. Furthermore, we say it occurs at the starting point of a generators m_i, a_j, b_j if it is located above, respectively, $s_{m_i}, s_{a_j}, s_{b_j}$ and at its endpoint if it is located above $t_{m_i}, t_{a_j}, t_{b_j}$. By comparing this assignment of intersection points via the graphical procedure with the assignment in theorem 3.2, we find that they agree for all $\lambda \in \pi_1(S_{g,n}\backslash D)$ with embedded representatives.

Theorem 3.4. *Consider an element $\lambda \in \pi_1(S_{g,n}\backslash D)$ with an embedded representative and given as a reduced word in the generators m_i, a_j, b_j. Then, the assignment of intersection points of λ with m_i, a_j, b_j between the different factors in this expression and to the starting points and endpoints of the generators m_i, a_j, b_j via the graphical procedure agrees with the one in theorem 3.2. In particular, the ambiguity for the assignment of an intersection point of λ with $x \in \{a_1, b_1, \ldots, a_g, b_g\}$ which arises at a factor $x^{\pm 1}$ in (55) corresponds to sliding the intersection point along x. In the following we assign such intersection points to the right of factors a_j, b_j and to the left of factors a_j^{-1}, b_j^{-1}.*

Proof. We first consider a single factor $y \in \{m_1^{\pm 1}, \ldots, b_g^{\pm 1}\}$ in the expression for λ as a reduced word in m_i, a_j, b_j and the intersection points of the associated curve with the representative of m_i, a_j, b_j.

(1) For each generator a_j, the graphical procedure implies that y has an intersection point with the starting point of a_j with positive intersection number if and only if $s_y \geqslant s_{a_j}$, which implies $y \in \{a_j^{\pm 1}, b_j^{\pm 1}, a_{j+1}^{\pm 1}, \ldots, b_g^{\pm 1}\}$. Similarly, it has an intersection point with the endpoint of a_j with negative intersection number if and only if $s_y \geqslant t_{a_j}$, $y \in \{a_j^{-1}, b_j^{-1}, a_{j+1}^{\pm 1}, \ldots, b_g^{\pm 1}\}$, and in both cases the intersection points occur on the segment $t_y p$. Intersection points at the starting point of a_j with negative intersection number and at the endpoint of a_j with positive intersection number lie on the segment ps_y and occur, respectively, for $t_y \geqslant s_{a_j}$, $y \in \{a_j^{\pm 1}, b_j^{\pm 1}, a_{j+1}^{\pm 1}, \ldots, b_g^{\pm 1}\}$, and $t_y \geqslant t_{a_j}$, $y \in \{a_j, b_j, a_{j+1}^{\pm 1}, \ldots, b_g^{\pm 1}\}$.

By comparing with expression (8) for the generators m_i, a_j, b_j in terms of their duals, we find that y gives rise to an intersection point at the starting point of a_j with positive and negative intersection number if and only if its expression as a reduced word in $\overline{m}_i, \overline{a}_j, \overline{b}_j$, respectively, ends in a sequence $\overline{a}_j\overline{h}_{j-1}\cdots\overline{m}_1$ and starts in a sequence $(\overline{a}_j\overline{h}_{j-1}\cdots\overline{m}_1)^{-1}$. Similarly, an intersection point of y with the endpoint of a_j with negative and positive intersection number occurs if and only if the expression for y ends in a sequence $\overline{a}_j^{-1}\overline{b}_j^{-1}\overline{a}_j\overline{h}_{j-1}\cdots\overline{m}_1$ and starts in a sequence $\left(\overline{a}_j^{-1}\overline{b}_j^{-1}\overline{a}_j\overline{h}_{j-1}\cdots\overline{m}_1\right)^{-1}$, respectively. Hence, each of the generators $\{a_{j+1}^{\pm 1}, \ldots, b_g^{\pm 1}\}$ has two intersections with each the starting and endpoint of a_j and with opposite intersection numbers, $b_j^{\pm 1}$ has two intersections with the starting point s_{a_j} with opposite intersection numbers and one with the endpoint t_{a_j}, where the oriented intersection number is positive for b_j and negative for b_j^{-1}. For the intersections of a_j with itself there is some ambiguity in drawing the associated intersection diagram. We can either assign two intersections with opposite intersection numbers to the starting point of a_i or one with positive intersection number to the endpoint and one with negative intersection number to the starting point. This corresponds to the fact that the two factors $\overline{a}_j, \overline{a}_j^{-1}$ in expression (8) for a_j can be considered either part of the sequence $\overline{m}_1^{-1}\cdots\overline{h}_{j-1}^{-1}\overline{a}_j^{-1}\overline{b}_j\overline{a}_j$ at the start of a_j or can be assigned to two sequences $\overline{m}_1^{-1}\cdots\overline{h}_{j-1}^{-1}\overline{a}_j^{-1}$ at the start of a_j and $\overline{a}_j\overline{h}_{j-1}\cdots\overline{m}_1$ at the end of a_j.

(2) Similarly, intersection points of y with the starting point of b_j with positive intersection number occur for $y \in \{b_j^{\pm 1}, a_j^{-1}, a_{j+1}^{\pm 1}, \ldots, b_g^{\pm 1}\}$ and those with the endpoint of b_j and negative intersection number for $y \in \{b_j, a_{j+1}^{\pm 1}, \ldots, b_g^{\pm 1}\}$ and both lie on the segment ps_y. Those at the starting point s_{b_j} with negative intersection number and at the endpoint t_{b_j} with positive intersection number are located on the segment $t_y p$ and occur for, respectively, $y \in \{b_j^{\pm 1}, a_j, a_{j+1}^{\pm 1}, \ldots, b_g^{\pm 1}\}$ and $y \in \{b_j^{-1}, a_{j+1}^{\pm 1}, \ldots, b_g^{\pm 1}\}$. Hence, intersection points with the starting point of b_j and, respectively, positive and negative intersection numbers can be identified with a sequence $\overline{b}_j^{-1}\overline{a}_j\overline{h}_{j-1}\cdots\overline{h}_1$ at the end of the expression for y as a reduced word in $\overline{m}_i, \overline{a}_j, \overline{b}_j$ and with a sequence $\left(\overline{b}_j^{-1}\overline{a}_j\overline{h}_{j-1}\cdots\overline{h}_1\right)^{-1}$ at its beginning. Intersection points at the end of b_j with, respectively, negative and positive intersection number correspond to sequences $\overline{h}_j\cdots\overline{h}_1$, $(\overline{h}_j\cdots\overline{h}_1)^{-1}$. Again, there is an ambiguity in the graphical assignment of the intersection points of b_j with itself, which can be either drawn as two intersection points at the starting point of b_j or as one at the starting point and one at the endpoint. This ambiguity corresponds to two different ways of assigning the two factors $\overline{b}_j, \overline{b}_j^{-1}$ in expression (8) for b_j.

(3) Finally, we consider intersection points of y with the generators m_i. We find that $y \in \{m_i^{-1}, m_{i+1}^{\pm 1}, \ldots, b_g^{\pm 1}\}$ has both an intersection point with the starting point of m_i and one with its endpoint, which lie on the segment ps_y and have, respectively, positive and negative intersection numbers. Similarly, $y \in \{m_i, m_{i+1}^{\pm 1}, \ldots, b_g^{\pm 1}\}$ has an intersection point with the starting point of m_i with negative intersection number and one with its endpoint with positive intersection number, both located on the segment ps_y. Hence, a pair of intersection points with the starting and endpoint of m_i and with opposite intersection numbers corresponds to a sequence $\overline{m}_i\cdots\overline{m}_1$ at the end of expression (8) for y if the one at the starting point has positive intersection number and to a sequence $(\overline{m}_i\cdots\overline{m}_1)^{-1}$ at the end of the expression if the one at the starting point has negative intersection number. Note that in contrast to the situation for the generators a_j, b_j there is no ambiguity in assigning the intersection points of m_i with itself since we required that one of the two intersection points lies at the starting point and one at the endpoint of m_i.

We now consider an element $\lambda \in \pi_1(S_{g,n}\setminus D)$ with an embedded representative and given as a reduced word in the generators m_i, a_j, b_j

$$\lambda = y_s^{\beta_s}\cdots y_1^{\beta_1} \qquad y_k \in \{m_1, \ldots, b_g\}, \quad \beta_k \in \{\pm 1\}. \tag{56}$$

In the graphical procedure, the intersection points of λ with m_i, a_j, b_j are obtained by decomposing it into its factors and removing those intersection points of the starting or endpoint of a generator m_i, a_j, b_j which occur both on a segment $t_{y_k}p$ and $ps_{y_{k+1}}$ with opposite intersection numbers. To minimize the number of intersection points, one makes use of the ambiguity in assigning the intersection points of a_j and b_j with themselves and removes the remaining ambiguity by assigning ambiguous intersection points to the right of a_j and b_j.

From the discussion above it follows that a pair of intersection points on the segments $t_{y_k^{\beta_k}}p$ and $ps_{y_{k+1}^{\beta_{k+1}}}$ can be removed if and only if the associated sequences $(\overline{m}_i\cdots\overline{m}_1)^{\pm 1}$, $(\overline{a}_j\overline{h}_{j_1}\cdots\overline{h}_1)^{\pm 1}$ $\left(\overline{a}_j^{-1}\overline{b}_j^{-1}\overline{a}_j\overline{h}_{j_1}\cdots\overline{h}_1\right)^{\pm 1}$, $\left(\overline{b}_j^{-1}\overline{a}_j\overline{h}_{j_1}\cdots\overline{h}_1\right)^{\pm 1}$, $\overline{h}_j\cdots\overline{h}_1)^{\pm 1}$ assigned to these intersection points as described above cancel. Factors $\overline{m}_i^{\pm 1}, \overline{a}_j^{\pm 1}, \overline{b}_j^{\pm 1}$ associated with intersection points of m_i, a_j, b_j with factors in (56) therefore give rise to factors $\overline{m}_i^{\pm 1}, \overline{a}_j^{\pm 1}, \overline{b}_j^{\pm 1}$ in the expression of λ as a reduced word in $\overline{m}_i, \overline{a}_j, \overline{b}_j$ if and only if the corresponding intersection points cannot be removed and remains in the intersection diagram. We then consider the precise form of these sequences in the generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$ and compare with the prescription in theorem 3.2. A short calculation using expressions (8) for the generators m_i, a_j, b_j and their duals then shows that the assignment of intersection points in theorem 3.2 agrees with the one from the graphical procedure for all intersection points that do not lie on the segment $t_s s_1$.

For intersection points on the segment $t_s s_1$, we note that the removal of intersection points by contracting the segments ps_1 and pt_s in the graphical procedure amounts to moving the basepoint of the curve representing λ. The number of resulting intersection points is minimal and cannot be reduced further by conjugating λ with elements of $\pi_1(S_{g,n}\setminus D)$. In the discussion after theorem 2.3, we found that this is the case if and only if the expression for λ as a reduced word in the generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$ is cyclically reduced. Hence, the assignment of intersection points obtained by graphically representing λ without contracting this segment agrees with the assignment in theorem 3.2. Contracting the segment $t_s s_1$ amounts to applying the procedure in theorem 3.2 to the cyclically reduced element associated with λ. □

The graphical procedure described above therefore reproduces the assignment of intersection points of $\lambda \in \pi_1(S_{g,n}\setminus D)$ with the generators m_i, a_j, b_j between the different factors in the expression of λ as a reduced word in m_i, a_j, b_j and to the starting and endpoint of the representatives of m_i, a_j, b_j in theorem 3.2. Furthermore, it allows us to consider general elements $\eta, \lambda \in \pi_1(S_{g,n}\setminus D)$ given as reduced words in the generators m_i, a_j, b_j and assign to their intersection points between the different factors in these expressions. For this, we represent both η and λ graphically as a product of curves representing m_i, a_j, b_j as described above. We draw two lines with starting and endpoints associated with the generators m_i, a_j, b_j, ordered according to (54) and basepoints p to the right of s_{m_i} such that the one for η is tangent to the boundary and the one for λ slightly displaced. After decomposing λ into a set of curves starting and ending above the corresponding starting and endpoints and into horizontal segments parallel to the line for λ, we can graphically determine the intersection points of λ with the generators m_i, a_j, b_j as described above. To obtain the intersection points of λ and η, we decompose η by consecutively drawing its factors in the expression as a reduced word in m_i, a_j, b_j and obtain a representative made up of curves starting and ending above the starting and endpoints on the line for η and of horizontal segments. All intersection points of the representatives of λ and η are then located above the starting and endpoints for η and on the horizontal segments in the decomposition of λ. Finally, one removes any intersection point which occurs both at the endpoint of a factor and at the starting point of the next factor in the decomposition of η with opposite intersection number by lifting the corresponding segment as shown in figure 9.

After completing this procedure, one obtains two curves representing η and λ with a minimum number of intersection points, all of which are located above the starting and endpoints on the line for η and on the horizontal segments for λ. By assigning an intersection point that occurs at the endpoint of a factor to the left of this factor and one that occurs at its starting point to the right, we then obtain a unique assignment of the intersection points of η and λ between the factors in the expression of η as a reduced word in m_i, a_j, b_j.

4. Dual generators and the moduli space of flat connections

4.1. Fock and Rosly's description of the moduli space of flat connections

In this section, we apply the involution $I \in \mathrm{Aut}(S_{g,n}\setminus D)$ to Fock and Rosly's description of the moduli space of flat connections on $S_{g,n}$ [3]. We show that by expressing the Poisson

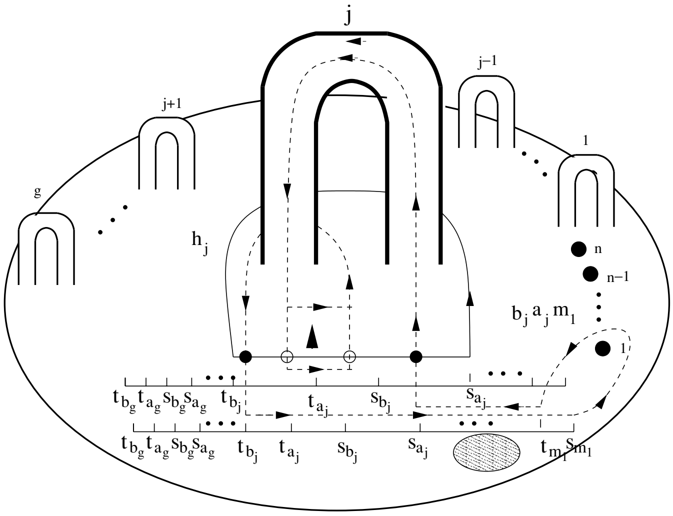

Figure 9. The intersection points of curves representing $\lambda = h_j = b_j \circ a_j^{-1} \circ b_j^{-1} \circ a_j$ (full line) and $\eta = b_j \circ a_j \circ m_1$ (dashed line).

structure on the moduli space in terms of both the generators $m_i, a_j, b_j \in \pi_1(S_{g,n}\setminus D)$ and their duals $\overline{m}_i, \overline{a}_j, \overline{b}_j$ one obtains a particularly simple expression in which its dependence on intersection points is apparent.

We start with a brief summary of moduli spaces of flat connections and Fock and Rosly's formalism. In the following we consider a finite-dimensional Lie group H with Lie algebra $\mathfrak{h} = \mathrm{Lie}H$, viewed as a vector space over $\mathbb{R}$. We fix a basis $J_a, a = 1, \ldots, \dim\mathfrak{h}$, of $\mathfrak{h}$ and denote by L_a, R_a, respectively, for the associated right- and left-invariant vector fields on H

$$L_a f(u) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(\mathrm{e}^{-tJ_a}u) \qquad R_a f(u) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(u\,\mathrm{e}^{tJ_a}) \qquad \forall u \in H, \forall f \in \mathcal{C}^\infty(H). \tag{57}$$

Here and in the following, we denote by $\exp : \mathfrak{h} \ni x^a J_a \mapsto \mathrm{e}^{x^a J_a} \in H$ the exponential map, which we require to be surjective.

The moduli space of flat H-connections on $S_{g,n}$ arises as the phase space of a Chern–Simons theory with gauge group H on the three-manifold $\mathbb{R} \times S_{g,n}$. To formulate this Chern–Simons theory, one associates to each puncture an orbit $\mathfrak{c}_i$ under the adjoint action of H on $\mathfrak{h}$

$$uJ_a u^{-1} = \mathrm{Ad}(u)_a{}^b J_b \qquad \forall u \in H \tag{58}$$

and fixes a non-degenerate, Ad-invariant, symmetric bilinear form $\langle , \rangle$ on $\mathfrak{h}$

$$\langle J_a, J_b\rangle = t_{ab} \qquad t_{ab}t^{bc} = \delta_a{}^c. \tag{59}$$

A flat H-connection on $S_{g,n}$ is a 1-form A on $S_{g,n}$ with values in the Lie algebra $\mathfrak{h}$ whose curvature F_A develops a delta-function singularity at each puncture and vanishes elsewhere

$$F_A = dA + A \wedge A = \sum_{i=1}^{n} T_i \delta(z - z(i)) \qquad T_i \in \mathfrak{c}_i, \tag{60}$$

where $z(i), i = 1, \ldots, n$, denotes the coordinate of the ith puncture. Gauge transformations are given by functions $\gamma : S_{g,n} \to H$ and act on the connection according to

$$A \mapsto \gamma A \gamma^{-1} + \gamma \, \mathrm{d}\gamma^{-1}. \tag{61}$$

The moduli space $\mathcal{M}^H_{g,n}$ is the quotient of the space of flat H-connections on $S_{g,n}$ modulo gauge transformations (61). Although defined as a quotient of an infinite-dimensional space, the moduli space $\mathcal{M}^H_{g,n}$ is finite dimensional and can be parametrized by the holonomies along a set of generators of the fundamental group $\pi_1(S_{g,n})$. While the holonomies $A_j = H_{a_j}, B_j = H_{b_j}$ of the generators associated with the handles are general elements of the gauge group H, the holonomies $M_i = H_{m_i}$ of the loops around the punctures are restricted to conjugacy classes $\mathcal{C}_i \subset H$ associated with the corresponding orbits $\mathfrak{c}_i$. Furthermore, the holonomies are subject to a constraint arising from the defining relation of the fundamental group $\pi_1(S_{g,n})$

$$\left[B_g, A_g^{-1}\right] \cdots \left[B_1, A_1^{-1}\right] \cdot M_n \cdots M_1 = 1 \qquad \left[B_j, A_j^{-1}\right] = B_j A_j^{-1} B_j^{-1} A_j. \tag{62}$$

Gauge transformations (61) act on the holonomies by simultaneously conjugating them with the elements of the gauge group H, and the moduli space $\mathcal{M}^H_{g,n}$ of flat H-connections on $S_{g,n}$ is given as the quotient of the holonomies modulo this simultaneous conjugation:

$$\begin{aligned}\mathcal{M}^H_{g,n} = \big\{&(M_1, \ldots, M_n, A_1, B_1, \ldots, A_g, B_g) \in H^{n+2g} | M_i \in \mathcal{C}_i, \\ &\left[B_g, A_g^{-1}\right] \cdots \left[B_1, A_1^{-1}\right] M_n \cdots M_1 = 1\big\} \big/ H.\end{aligned}$$

On the level of connections, the finite dimensionality of the moduli space $\mathcal{M}^H_{g,n}$ manifests itself in the fact that a flat H-connection can be trivialized, i.e. written as pure gauge, on any simply connected region $R \subset S_{g,n}$:

$$A|_R = \gamma \, \mathrm{d}\gamma^{-1} \qquad \text{with} \quad \gamma : R \to H. \tag{63}$$

Maximal simply connected regions are obtained by cutting the surface $S_{g,n}$ along a set of generators of the fundamental group. As in the case of a surface $S_{g,n} \setminus D$ discussed in section 2, this yields a set of n punctured discs and a polygon $P_{g,n}$, only that now the points x_0 and x_{n+4g} in figure 4 are identified since the boundary of the disc D is not present.

As shown by Alekseev and Malkin [14], a function $\gamma : P_{g,n} \to H$ defines a flat gauge field on $S_{g,n}$ if and only if it is such that the resulting holonomies around the punctures are elements of the conjugacy classes $\mathcal{C}_i$ and it satisfies an overlap condition for each pair of sides corresponding to a generator $y \in \{a_1, b_1, \ldots, a_g, b_g\}$:

$$A|_{y'} = \gamma \, \mathrm{d}\gamma^{-1}|_{y'} = \gamma \, \mathrm{d}\gamma^{-1}|_y = A|_y. \tag{64}$$

This requirement (64) is equivalent to the existence of constant elements $N_y \in H$ such that

$$\gamma^{-1}|_{y'} = N_y \gamma^{-1}|_y, \tag{65}$$

and the restriction of the holonomies around the punctures to conjugacy classes $\mathcal{C}_i$ can be cast into the form

$$\gamma^{-1}(x_i) = N_{m_i} \gamma^{-1}(x_{i-1}) \qquad N_{m_i}^{-1} \in \mathcal{C}_i. \tag{66}$$

The elements N_y, $y \in \{m_1, \ldots, m_n, a_1, b_1, \ldots, a_g, b_g\}$, in the overlap conditions (65), (66) contain all information about the physical state and are closely related to the holonomies M_i, A_j, B_j along our set of generators of the fundamental group $\pi_1(S_{g,n})$. It follows from figure 4 that these holonomies are given by the values of the trivializing function γ at the corners of the polygon:

$$\begin{aligned} M_i &= \gamma(x_i)\gamma^{-1}(x_{i-1}) \\ A_j &= \gamma(x_{n+4j-3})\gamma(x_{n+4j-4})^{-1} = \gamma(x_{n+4j-2})\gamma(x_{n+4j-1})^{-1} \\ B_j &= \gamma(x_{n+4j-3})\gamma(x_{n+4j-2})^{-1} = \gamma(x_{n+4j})\gamma(x_{n+4j-1})^{-1}. \end{aligned} \tag{67}$$

Using the overlap conditions (65), (66), we can express these holonomies in terms of the variables $N_{m_i}, N_{a_j}, N_{b_j} \in H$ and vice versa and obtain

$$\begin{aligned} N_{m_i} &= \gamma^{-1}(x_0)M_1^{-1}\cdots M_i^{-1}M_{i-1}\cdots M_1\gamma(x_0) \\ N_{a_j} &= \gamma^{-1}(x_0)M_1^{-1}\cdots M_n^{-1}H_1^{-1}\cdots H_j^{-1}B_jH_{j-1}\cdots H_1M_n\cdots M_1\gamma(x_0) \\ N_{b_j} &= \gamma^{-1}(x_0)M_1^{-1}\cdots M_n^{-1}H_1^{-1}\cdots H_j^{-1}A_jH_{j-1}\cdots H_1M_n\cdots M_1\gamma(x_0). \end{aligned} \tag{68}$$

Hence, up to conjugation with value of γ^{-1} at the basepoint x_0, the variables $N_{m_i}, N_{a_j}, N_{b_j}$ are the holonomies along the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$.

The moduli space $\mathcal{M}^H_{g,n}$ carries a canonical symplectic structure induced by the canonical symplectic form associated with the Chern–Simons action. An explicit and efficient description of the symplectic structure on the moduli space is provided by Fock and Rosly's formalism [3]. Fock and Rosly parametrize the symplectic structure on the moduli space in terms of an auxiliary Poisson structure on a finite-dimensional extended phase space, namely the space of graph connections associated with certain graphs on the surface $S_{g,n}$. After implementing a set of residual constraints which amount to a flatness condition on the graph connection and dividing by the associated graph gauge transformations, this auxiliary Poisson structure on the space of graph connections then induces the canonical Poisson structure on the moduli space $\mathcal{M}^H_{g,n}$.

In the following we will work with the formulation of Alekseev, Grosse and Schomerus [6–8] who specialized Fock and Rosly's description of the moduli space to the simplest graph describing the spatial surface $S_{g,n}$, a set of generators of its fundamental group $\pi_1(S_{g,n})$. In this case, the extended phase space is the manifold H^{n+2g}, and the different copies of H correspond to the holonomies along the generators of the fundamental group. Fock and Rosly's description of the Poisson structure on the moduli space can then be summarized as follows.

Theorem 4.1 (Fock, Rosly [3]). *Consider the manifold H^{n+2g} with points parametrized according to*

$$(M_1, \ldots, M_n, A_1, B_1, \ldots, A_g, B_g) \in H^{n+2g}, \tag{69}$$

and denote by L^X_a, R^X_a, $X \in \{M_1, \ldots, B_g\}$, the right- and left-invariant vector fields (56) associated with the different components of H^{n+2g}. Let $r = r^{ab}J_a \otimes J_b \in \mathfrak{h} \otimes \mathfrak{h}$ be a classical r-matrix for the Lie algebra $\mathfrak{h}$, i.e. a solution of the classical Yang–Baxter equation (CYBE)

$$\begin{aligned} &[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \\ &r_{12} := r^{ab}J_a \otimes J_b \otimes 1, r_{13} := r^{ab}J_a \otimes 1 \otimes J_b, r_{23} := r^{ab}1 \otimes J_a \otimes J_b, \end{aligned} \tag{70}$$

whose symmetric component is the dual of the bilinear form $\langle , \rangle$ in the Chern–Simons action:

$$r^{ab} = r^{ab}_{(s)} + r^{ab}_{(a)} \quad r^{ab}_{(a)} = \tfrac{1}{2}(r^{ab} - r^{ba}) \quad r^{ab}_{(s)} = \tfrac{1}{2}(r^{ab} + r^{ba}) = \tfrac{1}{2}t^{ab}. \tag{71}$$

Then, the Poisson bivector

$$
\begin{aligned}
B = r_{(a)}^{ab} & \left(\sum_{i=1}^{n} R_a^{M_i} + L_a^{M_i} + \sum_{j=1}^{g} R_a^{A_j} + L_a^{A_j} + R_a^{B_j} + L_a^{B_j}\right) \\
& \otimes \left(\sum_{i=1}^{n} R_b^{M_i} + L_b^{M_i} + \sum_{j=1}^{g} R_b^{A_j} + L_b^{A_j} + R_b^{B_j} + L_b^{B_j}\right) \\
& + \frac{1}{2} t^{ab} \left(\sum_{i=1}^{n} R_a^{M_i} + L_a^{M_i}\right) \wedge \left(\sum_{j=1}^{g} R_b^{A_j} + L_b^{A_j} + R_b^{B_j} + L_b^{B_j}\right) \\
& + \frac{1}{2} t^{ab} \sum_{i,j=1, i<j}^{n} \left(R_a^{M_i} + L_a^{M_i}\right) \wedge \left(R_b^{M_j} + L_b^{M_j}\right) \\
& + \frac{1}{2} t^{ab} \sum_{i,j=1, i<j}^{g} \left(R_a^{A_i} + L_a^{A_i} + R_a^{B_i} + L_a^{B_i}\right) \wedge \left(R_b^{A_j} + L_b^{A_j} + R_b^{B_j} + L_b^{B_j}\right) \\
& + \frac{1}{2} t^{ab} \sum_{i=1}^{n} R_a^{M_i} \wedge L_b^{M_i} + t^{ab} \sum_{i=1}^{g} R_a^{A_i} \wedge \left(R_b^{B_i} + L_b^{A_i} + L_b^{B_i}\right) \\
& + R_a^{B_i} \wedge \left(L_b^{A_i} + L_b^{B_i}\right) + L_a^{A_i} \wedge L_b^{B_i}
\end{aligned} \tag{72}
$$

defines a Poisson structure on H^{n+2g}. The symplectic structure on the moduli space $\mathcal{M}_{g,n}^H$ is obtained by restricting the components M_i to the conjugacy classes $\mathcal{C}_i$, by imposing the constraint (61) and by dividing by the associated gauge transformations which act by simultaneous conjugation of all components with H.

By realizing the moduli space of flat H-connections on $S_{g,n}$ as a quotient of the finite-dimensional Poisson manifold H^{n+2g}, Fock and Rosly's description of the moduli space [3] provides a rather efficient description of its Poisson structure. The Poisson bracket of functions $f \in \mathcal{C}^\infty(\mathcal{M}_{g,n}^H)$ on the moduli space $\mathcal{M}_{g,n}^H$ is given by the Fock–Rosly bracket (72) of the associated conjugation-invariant functions $f' \in \mathcal{C}^\infty(H^{n+2g})$:

$$
\left(\sum_{i=1}^{n} R_a^{M_i} + L_a^{M_i} + \sum_{j=1}^{g} R_a^{A_j} + L_a^{A_j} + R_a^{B_j} + L_a^{B_j}\right) f' = 0 \qquad a = 1, \ldots, \dim \mathfrak{h}. \tag{73}
$$

Note that although Fock and Rosly's formalism requires the choice of a classical r-matrix for the gauge group, the bracket of such conjugation-invariant functions with general functions $g \in \mathcal{C}^\infty(H^{n+2g})$ does not depend on the choice of the classical r-matrix. As the term involving its antisymmetric component $r_{(a)}$ in (72) vanishes if one of the functions is invariant under simultaneous conjugation, the resulting bracket depends only on the matrix t^{ab} representing the Ad-invariant bilinear form $\langle , \rangle$ in the Chern–Simons action.

A particular set of functions on the moduli space $\mathcal{M}_{g,n}^H$ is given by conjugation-invariant functions of the holonomies of closed curves of the surface $S_{g,n}$, which in the following will be referred to as generalized Wilson loop observables. As Fock and Rosly's Poisson structure is defined on the extended phase space H^{n+2g} where the constraint (62) from the defining relation of the fundamental group $\pi_1(S_{g,n})$ is not imposed, such functions are obtained from elements of the fundamental group $\pi_1(S_{g,n} \setminus D)$ of the associated surface with a disc removed.

More precisely, for each element $\lambda \in \pi_1(S_{g,n}\backslash D)$, given uniquely as a reduced word in the generators m_i, a_j, b_j

$$\lambda = x_r^{\alpha_r} \cdots x_1^{\alpha_1} \qquad x_k \in \{m_1, \ldots, b_g\}, \quad \alpha_k \in \{\pm 1\}, \tag{74}$$

one defines a map $\rho_\lambda : H^{n+2g} \to H$, which expresses the holonomy along λ in terms of the holonomies M_i, A_j, B_j along the generators of $\pi_1(S_{g,n}\backslash D)$

$$\rho_\lambda : (M_1, \ldots, M_n, A_1, B_1, \ldots, A_g, B_g) \mapsto H_\lambda = X_r^{\alpha_r} \cdots X_1^{\alpha_1} \quad X_k \in \{M_1, \ldots, B_g\}. \tag{75}$$

The generalized Wilson loop observables associated with λ are then obtained by composing conjugation-invariant functions $f \in \mathcal{C}^\infty(H)$ with the map ρ_λ:

$$f_\lambda = f \circ \rho_\lambda \in \mathcal{C}^\infty(H^{n+2g}). \tag{76}$$

As this map satisfies the condition

$$\rho_\lambda(uM_1u^{-1}, \ldots, uB_gu^{-1}) = u\rho_\lambda(M_1, \ldots, B_g)u^{-1}, \tag{77}$$

it follows immediately that the Wilson loop observables are invariant under simultaneous conjugation of all arguments with elements of the gauge group H and hence define a function on the moduli space $\mathcal{M}^H_{g,n}$:

$$\left(\sum_{i=1}^n R_a^{M_i} + L_a^{M_i} + \sum_{j=1}^g R_a^{A_j} + L_a^{A_j} + R_a^{B_j} + L_a^{B_j}\right) f_\lambda = (R_a + L_a) f \circ \rho_\lambda = 0$$
$$a = 1, \ldots, \dim \mathfrak{h}. \tag{78}$$

4.2. The Poisson structure in terms of the dual generators

The drawback of Fock and Rosly's description [3] of the Poisson structure on the moduli space is that it obscures the geometrical nature of the theory. For instance, it is known that the Poisson brackets of generalized Wilson loop observables depend on the intersection behaviour of the associated curves on the surface, i.e. the number of intersection points and the associated oriented intersection numbers. However, in Fock and Rosly's description of the Poisson structure on the moduli space in terms of the bivector (72) on the manifold H^{n+2g}, this dependence on intersection points is not readily apparent. In this subsection, we demonstrate that this problem can be remedied by working with the dual generators of the fundamental group. More precisely, we show that when expressed in terms of both the holonomies along the generators m_i, a_j, b_j and those along their duals $\overline{m}_i, \overline{a}_j, \overline{b}_j$, Fock and Rosly's Poisson structure takes a particularly simple form in which the dependence on intersection points becomes apparent.

For this, it is convenient to characterize Fock and Rosly's Poisson structure by the brackets of functions of the holonomies along our set of generators $m_i, a_j, b_j \in \pi_1(S_{g,n}\backslash D)$. Using the notation introduced in the last subsection, we denote by $f_\lambda \in \mathcal{C}^\infty(H^{n+2g})$ the function obtained by composing a general (not necessarily conjugation invariant) function $f \in \mathcal{C}^\infty(H)$ with the maps $\rho_\lambda : H^{n+2g} \to H$, $\lambda \in \pi_1(S_{g,n}\backslash D)$ as in (76). Using this notation, the Poisson bracket given by (72) can be expressed equivalently in terms of the functions $f_{m_i}, f_{a_j}, f_{b_j}$ as

$$\begin{aligned}\{f_x, g_x\} &= r^{ab}(R_a + L_a) f_x (R_b + L_b) g_x - t^{ab}(R_a + L_a) f_x R_b g_x \\ &= r^{ab}_{(a)}(R_a + L_a) f_x (R_b + L_b) g_x + \tfrac{1}{2} t^{ab}(R_a f_x L_b g_x - L_a f_x R_b g_x) \quad \forall x \in \{m_1, \ldots, b_g\}\end{aligned} \tag{79}$$

$$\begin{aligned}\{f_x, g_y\} &= r^{ab}(R_a + L_a) f_x (R_b + L_b) g_y \qquad \forall x, y \in \{m_1, \ldots, b_g\}, x < y \\ &= r^{ab}_{(a)}(R_a + L_a) f_x (R_b + L_b) g_y + \tfrac{1}{2} t^{ab}(R_a + L_a) f_x (R_b + L_b) g_y\end{aligned} \tag{80}$$

$$\{f_{a_j}, g_{b_j}\} = r^{ab}(R_a + L_a) f_{a_j}(R_b + L_b) g_{b_j} - t^{ab} L_a f_{a_j} R_b g_{b_j}$$
$$= r^{ab}_{(a)}(R_a + L_a) f_{a_j}(R_b + L_b) g_{b_j} + \tfrac{1}{2} t^{ab}(R_a f_{a_j} L_b g_{b_j} - L_a f_{a_j} R_b g_{b_j}), \quad (81)$$

where L_a, R_a denote the right- and left-invariant vector fields (57) on H and $<$ in (80) stands for the ordering

$$x < y \Leftrightarrow \begin{cases} x = m_i,\ y = m_j,\ i, j \in \{1, \dots, n\},\ i < j \\ x \in \{m_1, \dots, m_n\},\ y \in \{a_1, b_1, \dots, a_g, b_g\} \\ x \in \{a_i, b_i\},\ y \in \{a_j, b_j\},\ i, j \in \{1, \dots, g\},\ i < j. \end{cases} \quad (82)$$

By using the expressions (8) for the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$ in terms of m_i, a_j, b_j, we can derive the Poisson brackets of functions $f_{m_i}, f_{a_j}, f_{b_j}$ of the holonomies along our generators with functions $g_{\overline{m}_i}, g_{\overline{a}_j}, g_{\overline{b}_j}$ of the holonomies along their duals. A somewhat lengthy but straightforward calculation using the identities (8), (72) and the Ad-invariance of the bilinear form $\langle , \rangle$ then yields the following theorem.

Theorem 4.2. *Consider functions $f, g \in \mathcal{C}^\infty(H)$ and the associated functions $f_{m_i}, f_{a_j}, f_{b_j} \in \mathcal{C}^\infty(H^{n+2g})$, $g_{\overline{m}_i}, g_{\overline{a}_j}, g_{\overline{b}_j} \in \mathcal{C}^\infty(H^{n+2g})$ of the holonomies along the generators $m_i, a_j, b_j \in \pi_1(S_{g,n} \setminus D)$ and their duals defined as in (75). Then, Fock and Rosly's Poisson bracket on the manifold H^{n+2g} is characterized by the following brackets:*

$$\{f_x, g_{\overline{y}}\} = -r^{ba}(R_a + L_a) f_x (R_b + L_b) g_{\overline{y}} \quad \forall x \in \{m_1, \dots, b_g\}, \overline{y} \in \{\overline{m}_1, \dots, \overline{b}_g\}, \quad x \neq y \quad (83)$$

$$\{f_{m_i}, g_{\overline{m}_i}\} = -r^{ba}(R_a + L_a) f_{m_i}(R_b + L_b) g_{\overline{m}_i} - t^{ab}(R_a + L_a) f_{m_i} \mathrm{Ad}(M_i \cdots M_1)_b{}^c L_c g_{\overline{m}_i}$$
$$= -r^{ba}(R_a + L_a) f_{m_i}(R_b + L_b) g_{\overline{m}_i} + t^{ab} R_a f_{m_i} \mathrm{Ad}(M_{i-1} \cdots M_1)_b{}^c (L_c + R_c) g_{\overline{m}_i} \quad (84)$$

$$\{f_{a_j}, g_{\overline{a}_j}\} = -r^{ba}(R_a + L_a) f_{a_j}(R_b + L_b) g_{\overline{a}_j} - t^{ab} R_a f_{a_j} \mathrm{Ad}\big(B_j{}^{-1} H_j \cdots H_1 M_n \cdots M_1\big)_b{}^c L_c g_{\overline{a}_j}$$
$$= -r^{ba}(R_a + L_a) f_{a_j}(R_b + L_b) g_{\overline{a}_j} + t^{ab} R_a f_{a_j} \mathrm{Ad}(H_{j-1} \cdots H_1 M_n \cdots M_1)_b{}^c R_c g_{\overline{a}_j} \quad (85)$$

$$\{f_{b_j}, g_{\overline{b}_j}\} = -r^{ba}(R_a + L_a) f_{b_j}(R_b + L_b) g_{\overline{b}_j} + t^{ab} R_a f_{b_j} \mathrm{Ad}\big(B_j{}^{-1} H_j \cdots H_1 M_n \cdots M_1\big)_b{}^c L_c g_{\overline{b}_j}$$
$$= -r^{ba}(R_a + L_a) f_{b_j}(R_b + L_b) g_{\overline{b}_j} - t^{ab} R_a f_{b_j} \mathrm{Ad}\big(B_j{}^{-1} A_j H_{j-1} \cdots H_1 M_n \cdots M_1\big)_b{}^c R_c g_{\overline{b}_j}. \quad (86)$$

In expressions (83)–(86), the Poisson brackets of the functions $f_x, g_{\overline{y}} \in \mathcal{C}^\infty(H^{n+2g})$ are given as a sum of a global conjugation term involving the classical r-matrix and of a term which depends only on the components t^{ab} of the bilinear form in the Chern–Simons action. The former vanishes if one of the two functions is conjugation invariant, i.e. represents a function on the moduli space $\mathcal{M}^H_{g,n}$. The latter is nontrivial only in the brackets of functions of a generator $x \in \{m_1, \dots, b_g\}$ with functions of its dual $\overline{x}$. This reflects the fact that the Wilson loop observables associated with different curves on the spatial surface have non-vanishing Poisson brackets only if these curves intersect. As shown in the previous section, the intersection points of a general curve $\lambda \in \pi_1(S_{g,n} \setminus D)$ with the generators m_i, a_j, b_j correspond to factors $\overline{m}_i^{\pm 1}, \overline{a}_j^{\pm 1}, \overline{b}_j^{\pm 1}$ in the expression of λ as a reduced word in the dual generators. Formula (83) therefore implies that each intersection point of a general embedded curve $\lambda \in \pi_1(S_{g,n} \setminus D)$ with the generators a_j, b_j and each pair of intersection points of λ with a generator m_i give rise to a summand in the Poisson brackets of a generalized Wilson loop observable associated with λ with functions of the holonomies along the generators m_i, a_j, b_j. We will investigate this dependence on intersection points in more detail in section 5, where we derive a formula for the Poisson brackets of generalized Wilson loop observables with general functions $f \in \mathcal{C}^\infty(H^{n+2g})$.

To obtain a more general formulation which clarifies the role of the involution $I \in \mathrm{Aut}(\pi_1(S_{g,n}\backslash D))$ in the description of the moduli space, we consider the diffeomorphism $\Phi_I : H^{n+2g} \to H^{n+2g}$ induced by I. This diffeomorphism maps the components of H which represent the holonomies along the generators m_i, a_j, b_j to the holonomies along their duals:

$$\begin{aligned}
&\Phi_I : (M_1, \dots, M_n, A_1, B_1, \dots, A_g, B_g) \to (\overline{M}_1, \dots, \overline{M}_n, \overline{A}_1, \overline{B}_1, \dots, \overline{A}_g, \overline{B}_g)\\
&\overline{M}_i = M_1^{-1} \cdots M_i^{-1} M_{i-1} \cdots M_1\\
&\overline{A}_j = M_1^{-1} \cdots H_j^{-1} B_j H_{j-1} \cdots M_1\\
&\overline{B}_j = M_1^{-1} \cdots H_j^{-1} A_j H_{j-1} \cdots M_1.
\end{aligned} \tag{87}$$

More generally, for any $\lambda \in \pi_1(S_{g,n}\backslash D)$ with dual $I(\lambda)$, the holonomy along $I(\lambda)$ is obtained by composing the map $\rho_\lambda : H^{n+2g} \to H$ with Φ_I:

$$\rho_{I(\lambda)} = \rho_\lambda \circ \Phi_I \qquad \forall \lambda \in \pi_1(S_{g,n}\backslash D). \tag{88}$$

Using this identity together with expressions (83)–(86) for the Poisson bracket, we find that the bracket of any function $f \in \mathcal{C}^\infty(H^{n+2g})$ which is invariant under simultaneous conjugation and general $g \in \mathcal{C}^\infty(H^{n+2g})$ takes the form

$$\begin{aligned}
\{f, g \circ \Phi_I\} = {} & t^{ab} \sum_{i=1}^{n} R_a^{M_i} f \,\mathrm{Ad}(M_{i-1} \cdots M_1)_b{}^c \big(\big(R_c^{M_i} + L_c^{M_i}\big) g\big) \circ \Phi_I\\
& + t^{ab} \sum_{j=1}^{g} R_a^{A_j} f \,\mathrm{Ad}(H_{j-1} \cdots M_1)_b{}^c \big(R_c^{A_j} g\big) \circ \Phi_I\\
& - t^{ab} \sum_{j=1}^{g} R_a^{B_j} f \,\mathrm{Ad}(B_j^{-1} A_j H_{j-1} \cdots M_1)_b{}^c \big(R_c^{B_j} g\big) \circ \Phi_I.
\end{aligned} \tag{89}$$

To derive a general formula for the transformation of the Poisson structure under the involution $I \in \mathrm{Aut}(S_{g,n}\backslash D)$ and the associated diffeomorphism $\Phi_I : H^{n+2g} \to H^{n+2g}$, we express Fock and Rosly's Poisson structure entirely in terms of functions $f_{\overline{m}_i}, f_{\overline{a}_j}, f_{\overline{b}_j}$ associated with the holonomies along the dual generators $\overline{m}_i, \overline{a}_i, \overline{b}_i \in \pi_1(S_{g,n}\backslash D)$. Using the expressions (8) for $\overline{m}_i, \overline{a}_j, \overline{b}_j$ in terms of the original generators m_i, a_j, b_j and formulae (83)–(86) for the Poisson bracket, we obtain

$$\begin{aligned}
\{f_{\overline{x}}, g_{\overline{x}}\} &= -r^{ba}(R_a + L_a) f_{\overline{x}} (R_b + L_b) g_{\overline{x}} + t^{ab}(R_a + L_a) f_{\overline{x}} R_b g_{\overline{x}}\\
&= r_{(a)}^{ab}(R_a + L_a) f_{\overline{x}} (R_b + L_b) g_{\overline{x}} - \tfrac{1}{2} t^{ab}(R_a f_{\overline{x}} L_b g_{\overline{x}} - L_a f_{\overline{x}} R_b g_{\overline{x}})\\
&\quad \forall \overline{x} \in \{\overline{m}_1, \dots, \overline{b}_g\}
\end{aligned} \tag{90}$$

$$\begin{aligned}
\{f_{\overline{x}}, g_{\overline{y}}\} &= -r^{ba}(R_a + L_a) f_{\overline{x}} (R_b + L_b) g_{\overline{y}} \qquad \forall \overline{x}, \overline{y} \in \{\overline{m}_1, \dots, \overline{b}_g\}, \quad \overline{x} < \overline{y}\\
&= r_{(a)}^{ab}(R_a + L_a) f_{\overline{x}} (R_b + L_b) g_{\overline{y}} - \tfrac{1}{2} t^{ab}(R_a + L_a) f_{\overline{x}} (R_b + L_b) g_{\overline{y}}
\end{aligned} \tag{91}$$

$$\begin{aligned}
\{f_{\overline{a}_j}, g_{\overline{b}_j}\} &= -r^{ba}(R_a + L_a) f_{\overline{a}_j} (R_b + L_b) g_{\overline{b}_j} + t^{ab} L_a f_{\overline{a}_j} R_b g_{\overline{b}_j}\\
&= r_{(a)}^{ab}(R_a + L_a) f_{\overline{a}_j} (R_b + L_b) g_{\overline{b}_j} - \tfrac{1}{2} t^{ab}(R_a f_{\overline{a}_j} L_b g_{\overline{b}_j} - L_a f_{\overline{a}_j} R_b g_{\overline{b}_j}),
\end{aligned} \tag{92}$$

where the ordering in (91) is the one obtained by replacing each generator in the ordering (82) with its dual. By comparing these brackets with expressions (81)–(90) for the Fock–Rosly Poison brackets of the functions $f_{m_i}, f_{a_j}, f_{b_j}$ associated with the original generators, we find that they take the same form up to a flip and a sign change in the classical r-matrix and obtain the following theorem.

Theorem 4.3. *The Fock–Rosly Poisson bivector (72) is form invariant under the simultaneous exchange of the generators $m_i, a_j, b_j \in \pi_1(S_{g,n}\setminus D)$ with their duals $\overline{m}_i, \overline{a}_j, \overline{b}_j \in \pi_1(S_{g,n}\setminus D)$ and of the r-matrix $r = r^{ab} J_a \otimes J_b$ with its flip $-\sigma(r) = -r^{ba} J_a \otimes J_b$:*

$$\{f \circ \Phi_I, g \circ \Phi_I\}_r = \{f, g\}_{-\sigma(r)} \circ \Phi_I \qquad \forall f, g \in \mathcal{C}^\infty(H^{n+2g}). \tag{93}$$

In particular, for any $f \in \mathcal{C}^\infty(H^{n+2g})$ invariant under simultaneous conjugation and arbitrary $g \in \mathcal{C}^\infty(H^{n+2g})$ we have

$$\{f \circ \Phi_I, g \circ \Phi_I\} = -\{f, g\} \circ \Phi_I. \tag{94}$$

As the Poisson structure on the moduli space $\mathcal{M}^H_{g,n}$ is given by the Fock–Rosly Poisson brackets of conjugation-invariant functions on H^{n+2g} which do not depend on the antisymmetric part of the r-matrix but only the components t^{ab} of the Ad-invariant symmetric bilinear form, this implies that the Poisson structure on the moduli space is invariant under an exchange of the generators m_i, a_j, b_j and their duals up to a global minus sign. We will demonstrate in section 6 that this is the case for any automorphism of $\pi_1(S_{g,n}\setminus D)$ which satisfies the condition (4) with $w = 1, \epsilon = -1$.

5. Application: the phase space transformations generated by Wilson loop observables

5.1. The Poisson brackets of Wilson loop observables

In this section, we use the dual generators of the fundamental group to derive explicit expressions for the Poisson bracket of generalized Wilson loop observables associated with embedded curves on the surface $S_{g,n}\setminus D$ and to determine the associated flows on the extended phase space H^{n+2g}. For the case of a surface without punctures, the Poisson brackets of generalized Wilson loop observables and the associated flows on the moduli space $\mathcal{M}^H_{g,0}$ were first determined by Goldman [11] who uses cohomological methods and characterizes these quantities in terms of the intersection behaviour of the associated curves on $S_{g,n}$. The dual generators of the fundamental group allow us to generalize these results to punctured surfaces. Moreover, we obtain a purely algebraic formulation, in which these flows are characterized by the transformation of the holonomies along our set of generators of the fundamental group $\pi_1(S_{g,n}\setminus D)$ and derived from the expression of the associated curves as reduced words in the dual generators.

The first step is to determine the Poisson bracket of a general function $g \in \mathcal{C}^\infty(H^{n+2g})$ with the Wilson loop observable f_λ associated with a conjugation-invariant function $f \in \mathcal{C}^\infty(H)$ and with an element $\lambda \in \pi_1(S_{g,n}\setminus D)$ with an embedded representative. To calculate the bracket $\{g, f_\lambda\}$, one inserts the expression (97) of λ as a reduced word in the generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$ into formula (89) for the Poisson bracket. As the maps $\rho_\lambda : H^{n+2g} \to H$ satisfy the identity

$$\rho_{\tau\circ\lambda\circ\tau^{-1}} = \rho_\tau \cdot \rho_\lambda \cdot \rho_\tau^{-1}, \tag{95}$$

we have for any conjugation-invariant function $f \in \mathcal{C}^\infty(H)$

$$f(\rho_\lambda(u) \cdot \rho_\tau^{-1}(u) g \rho_\tau(u)) = f(\rho_{\tau\circ\lambda\circ\tau^{-1}}(u) g) \qquad \forall g \in H, \quad u \in H^{n+2g}. \tag{96}$$

By applying this identity together with the expressions (8) to the terms in (89) involving the adjoint action, we then obtain the following theorem.

Theorem 5.1. *Consider an element $\lambda \in \pi_1(S_{g,n}\backslash D)$ with an embedded representative given uniquely as a reduced word in the generators $\overline{m}_i, \overline{a}_j, \overline{b}_j \in \pi_1(S_{g,n}\backslash D)$ by*

$$\lambda = \overline{x}_r^{\alpha_r} \cdots \overline{x}_1^{\alpha_1} \qquad \overline{x}_k \in \{\overline{m}_1, \ldots, \overline{b}_g\}, \quad \alpha_k \in \{\pm 1\} \tag{97}$$

and let $f \in \mathcal{C}^\infty(H)$ be conjugation invariant. Then, the Poisson bracket of a general function $g \in \mathcal{C}^\infty(H^{n+2g})$ with the gauge-invariant observable $f_\lambda = f \circ \rho_\lambda$ is given by

$$\begin{aligned}
\{g, f_\lambda\} = \sum_{i=1}^{n} t^{ab}\big(R_a^{M_i} g + L_a^{M_i} g\big) & \Bigg(\sum_{\overline{x}_k = \overline{m}_i, \alpha_k = 1} R_b f_{\mathrm{Ad}(m_{i-1}\cdots m_1 \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1})\lambda} \\
&\quad - \sum_{\overline{x}_k = \overline{m}_i, \alpha_k = -1} R_b f_{\mathrm{Ad}(m_i \cdots m_1 \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1})\lambda}\Bigg) \\
&+ \sum_{j=1}^{g} t^{ab} R_a^{A_j} g \Bigg(\sum_{\overline{x}_k = \overline{a}_j, \alpha_k = 1} R_b f_{\mathrm{Ad}(h_{j-1}\cdots m_1 \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1})\lambda} \\
&\quad - \sum_{\overline{x}_k = \overline{a}_j, \alpha_k = -1} R_b f_{\mathrm{Ad}(a_j^{-1} b_j^{-1} a_j h_{j-1} \cdots m_1 \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1})\lambda}\Bigg) \\
&- \sum_{j=1}^{g} t^{ab} R_a^{B_j} g \Bigg(\sum_{\overline{x}_k = \overline{b}_j, \alpha_k = 1} R_b f_{\mathrm{Ad}(b_j^{-1} a_j h_{j-1} \cdots m_1 \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1})\lambda} \\
&\quad - \sum_{\overline{x}_k = \overline{b}_j, \alpha_k = -1} R_b f_{\mathrm{Ad}(a_j^{-1} b_j a_j h_{j-1} \cdots m_1 \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1})\lambda}\Bigg),
\end{aligned} \tag{98}$$

where we write $f_{\mathrm{Ad}(\tau)\lambda} = f \circ \rho_{\tau\circ\lambda\circ\tau^{-1}}$.

Note that both, the observable $f_{\mathrm{Ad}(\tau)\lambda} = f_\lambda$ and the right-hand side of (98), are invariant under conjugation $\lambda \to \tau \circ \lambda \circ \tau^{-1}$ with a general element $\tau \in \pi_1(S_{g,n}\backslash D)$. Although the factors $\overline{m}_i, \overline{a}_j, \overline{b}_j$ in the decomposition of τ give rise to additional summands in (98), their contributions cancel pairwise. Conversely, two summands in (98) which cancel each other can arise only if λ is of the form $\lambda = \tau \circ \tilde{\lambda} \circ \tau^{-1}$, $\tilde{\lambda} \in \pi_1(S_{g,n}\backslash D)$, and both of them result from factors in the decomposition of τ. As the Poisson bracket of f_λ depends only on the conjugacy class $[\lambda] = \{\tau\lambda\tau^{-1} | \tau \in \pi_1(S_{g,n}\backslash D)\}$, we can simplify calculations by restricting attention to curves $\lambda \in \pi_1(S_{g,n}\backslash D)$ whose expression as a reduced word in $\overline{m}_i, \overline{a}_j, \overline{b}_j$ is also cyclically reduced. Hence, we have obtained an explicit expression for the Poisson bracket of the gauge-invariant observable f_λ with a general function on H^{n+2g} in terms of functions associated with certain elements in the conjugacy class of λ. To achieve a more geometrical interpretation of formula (98), we note that the summands in (98) are in one-to-one correspondence with factors $\overline{x}_k = \overline{m}_i, \overline{a}_j, \overline{b}_j$ in the decomposition (97) of λ and their signs—modulo an overall sign for generators $\overline{b}_j$—are given by the corresponding exponents α_k. In section 2, we showed that factors $\overline{x}_k = \overline{m}_i, \overline{a}_j, \overline{b}_j$ in the expression (15) correspond to, respectively, intersection points of λ with the generators m_i, a_j, b_j and their exponent α_k determines the oriented intersection number. Furthermore, the factors τ in expressions of the form $f_{\mathrm{Ad}(\tau)\lambda}$ in (98) are precisely the ones in expressions (30), (31), (33), (34) in theorem 3.2, which give the splitting of the generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$ as reduced words in m_i, a_j, b_j to assign these intersection points between the different factors in the expression of λ. Also note that the ambiguity in moving these intersection points to either the starting point or endpoint of the generators a_j, b_j which we encountered in section 3.2 is reflected in formula (98). Using the Ad-invariance of the bilinear form $\langle , \rangle$, the conjugation invariance of f and the identity (95), we find

$$t^{ab} R^a_X g R^b f_{\mathrm{Ad}(\tau)\lambda} = -t^{ab} L^a_X g R^b f_{\mathrm{Ad}(x\tau)\lambda} \qquad \forall x \in \{a_1, b_1, \ldots, a_g, b_g\}, \quad g \in \mathcal{C}^\infty(H^{n+2g}), \tag{99}$$

which corresponds to shifting an intersection point at the starting point of $x \in \{a_1, b_1, \ldots, a_g, b_g\}$ to its endpoint. By applying this identity to all factors $\overline{x}_k^{\alpha_k} \in \{\overline{a}_1, \overline{b}_1, \ldots, \overline{a}_g, \overline{b}_g\}$ where the corresponding intersection point is located at the endpoint of a generator a_j, b_j, we reproduce the assignment in theorem 3.2 and obtain the following.

Corollary 5.2. *Consider an element $\lambda \in \pi_1(S_{g,n}\backslash D)$ with an embedded representative and given uniquely as a product in the generators $m_i, a_j, b_j \in \pi_1(S_{g,n}\backslash D)$ and their duals by*

$$\lambda = \overline{x}_r^{\alpha_r} \cdots \overline{x}_1^{\alpha_1} = z_t^{\delta_t} \circ \ldots \circ z_1^{\delta_1} \qquad x_i, z_j \in \{m_1, \ldots, b_g\}, \quad \alpha_i, \delta_j \in \{\pm 1\}. \tag{100}$$

Assign the intersection points of λ with m_i, a_j, b_j between the different factors $z_k^{\delta_k}$ in (100) and to the starting and endpoints of the generators m_i, a_j, b_j as in theorem 3.2. Then, the Poisson bracket (98) can be written as

$$\begin{aligned}
\{g, f_\lambda\} = \sum_{i=1}^n t^{ab} \Bigg(& \sum_{k: t_k s_{k+1} \cap m_i = s_{m_i}} \epsilon(m_i, t_k s_{k+1}) R_a^{M_i} g R_b f_{\mathrm{Ad}(z_k^{\delta_k} \cdots z_1^{\delta_1})\lambda} \\
& - \sum_{k: t_{k+1} s_k \cap m_i = t_{m_i}} \epsilon(m_i, t_k s_{k+1}) L_a^{M_i} g R_b f_{\mathrm{Ad}(z_k^{\delta_k} \cdots z_1^{\delta_1})\lambda} \Bigg) \\
& + \sum_{j=1}^g t^{ab} \Bigg(\sum_{k: t_k s_{k+1} \cap a_j = s_{a_j}} \epsilon(a_j, t_k s_{k+1}) R_a^{A_j} g R_b f_{\mathrm{Ad}(z_k^{\delta_k} \cdots z_1^{\delta_1})\lambda} \\
& - \sum_{k: t_{k+1} s_k \cap a_j = t_{a_j}} \epsilon(a_j, t_k s_{k+1}) L_a^{A_j} g R_b f_{\mathrm{Ad}(z_k^{\delta_k} \cdots z_1^{\delta_1})\lambda} \Bigg) \\
& + \sum_{j=1}^g t^{ab} \Bigg(\sum_{k: t_k s_{k+1} \cap b_j = s_{b_j}} \epsilon(b_j, t_k s_{k+1}) R_a^{B_j} g R_b f_{\mathrm{Ad}(z_k^{\delta_k} \cdots z_1^{\delta_1})\lambda} \\
& - \sum_{k: t_{k+1} s_k \cap b_j = t_{b_j}} \epsilon(b_j, t_k s_{k+1}) L_a^{B_j} g R_b f_{\mathrm{Ad}(z_k^{\delta_k} \cdots z_1^{\delta_1})\lambda} \Bigg),
\end{aligned} \tag{101}$$

where $\epsilon(x, t_k s_{k+1})$ stands for the oriented intersection number of $x \in \{m_1, \ldots, b_g\}$ with the oriented segment $t_k s_{k+1}$, and we write $t_k s_{k+1} \cap x = s_x$ if the intersection point of the oriented segment $t_k s_{k+1}$ with x is located at the starting point of x and $t_k s_{k+1} \cap x = t_x$ if it is located at the endpoint.

Corollary 5.2 allows us to derive an explicit expression for the Poisson bracket of the Wilson loop observables associated with elements $\lambda, \eta \in \pi_1(S_{g,n}\backslash D)$ with embedded representatives. By applying formula (101) to the Wilson loop observable g_η associated with a conjugation-invariant function $g \in \mathcal{C}^\infty(H)$ and $\eta \in \pi_1(S_{g,n}\backslash D)$, we find that if a given segment $t_k s_{k+1}$ intersects a factor in the expression for η as a reduced word in m_i, a_j, b_j at its endpoint and the next factor in its starting point with opposite intersection number, the contributions of these factors cancel. Recalling the discussion after theorem 3.4, we note that this corresponds to the graphical procedure for removing unnecessary intersection points of the curves representing η and λ. Using the identity (99), we can express all vector fields acting on g_η in terms of the left-invariant vector fields for the different arguments and transform them into expressions of the form $g_{\mathrm{Ad}(\tau)\eta}$ via (96). We obtain the following corollary.

Corollary 5.3. *Consider conjugation-invariant functions $f, g \in \mathcal{C}^\infty(H)$ and elements $\eta, \lambda \in \pi_1(S_{g,n} \setminus D)$ with embedded representatives and given uniquely as reduced words in the generators m_i, a_j, b_j by*

$$\eta = y_s^{\beta_s} \circ \cdots \circ y_1^{\beta_1} \qquad \lambda = z_t^{\delta_t} \circ \ldots \circ z_1^{\delta_1} \qquad y_i, z_j \in \{m_1, \ldots, b_g\}, \quad \beta_i, \delta_j \in \{\pm 1\}. \tag{102}$$

Let p_k, $k = 1, \ldots, m$, denote the intersection points of λ and η such that p_k occurs between the factors $y_{i_k}^{\beta_{i_k}}$ and $y_{i_k+1}^{\beta_{i_k+1}}$ in the expression (102) of η and between the factors $z_{j_k}^{\delta_{j_k}}$ and $z_{j_k+1}^{\delta_{j_k+1}}$ in the expression of λ and denote by $\epsilon_k(\eta, \lambda) = \epsilon(\eta, \lambda, p_k)$ the oriented intersection number of η and λ in p_k. Then, the Poisson bracket of the observables g_η and f_λ is given by

$$\{g_\eta, f_\lambda\} = \sum_{k=1}^{m} \epsilon_k(\eta, \lambda) t^{ab} (R_a g)_{(y_{i_k}^{\beta_{i_k}} \cdots y_1^{\beta_1}) \circ \eta \circ (y_{i_k}^{\beta_{i_k}} \cdots y_1^{\beta_1})^{-1}} (R_b f)_{(z_{j_k}^{\delta_{j_k}} \cdots z_1^{\delta_1}) \circ \lambda \circ (z_{j_k}^{\delta_{j_k}} \cdots z_1^{\delta_1})^{-1}}. \tag{103}$$

For surfaces $S_{g,0}$ without punctures, formula (103) gives an algebraic version of Goldman's product formula, see theorem 3.5. in [11]. Moreover, it generalizes this formula to punctured surfaces not considered in [11]. The concept of duality for a set of generators of the fundamental group therefore establishes a link between the purely algebraic description of the Poisson brackets of generalized Wilson loop observables and the geometrical formulation in terms of intersection points derived in [11]. We will find in the next subsection that this description can be used to obtain an algebraic formula of the associated flows on phase space.

5.2. The flows generated by the Wilson loop observables: the geometrical formulation

After determining the Poisson bracket of generalized Wilson loop observables with general functions $g \in \mathcal{C}^\infty(H^{n+2g})$, we will now derive the associated one-parameter groups of diffeomorphisms of H^{n+2g} these observables generate via the Poisson bracket.

In the following we will often parametrize elements of the group H in terms of the exponential map $\exp : \mathfrak{h} \to H$. While we require this map to be surjective throughout this paper, we will not suppose that it is injective. This implies that the exponential map is locally but not globally bijective and that the parametrization of group elements in terms of Lie algebra elements is in general not unique. Following Goldman [11], we define for any conjugation-invariant function $f \in \mathcal{C}^\infty(H)$ a Lie algebra valued map $g_f : H \to \mathfrak{h}$ and an associated one-parameter group of diffeomorphisms $G_f^t : H \to H$:

$$\langle g_f(u), J_a \rangle = R_a f(u) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(u\, \mathrm{e}^{tJ_a}) \quad G_f^t(u) = \mathrm{e}^{t g_f(u)} \qquad \forall u \in H, a = 1, \ldots, \dim \mathfrak{h}, \tag{104}$$

where the parameter $t \in \mathbb{R}$ is restricted appropriately to ensure the bijectivity of the exponential map. As the bilinear form $\langle , \rangle$ is non-degenerate, equation (104) defines g_f uniquely as

$$g_f(u) = t^{ab} R_a f(u) J_b, \tag{105}$$

and from the Ad-invariance of the bilinear form $\langle , \rangle$ it follows that the Lie algebra valued functions $g_f : H \to \mathfrak{h}$ and the associated diffeomorphisms $G^t_f : H \to H$ satisfy the covariance conditions

$$g_f(gug^{-1}) = g g_f(u) g^{-1} \qquad G^t_f(gug^{-1}) = g G^t_f(u) g^{-1} \qquad \forall g, u \in H. \tag{106}$$

For the functions

$$G^t_{f,\lambda} = G^t_f \circ \rho_\lambda : H^{n+2g} \to H, \tag{107}$$

obtained by composing these diffeomorphisms with the map $\rho_\lambda : H^{n+2g} \to H$ to the holonomy along λ, this covariance condition and the identity (95) imply

$$G^t_{f,\tau\circ\lambda\circ\tau^{-1}} = \rho_\tau \cdot G^t_{f,\lambda} \cdot \rho_\tau^{-1}. \tag{108}$$

We now consider an element $\lambda \in \pi_1(S_{g,n} \setminus D)$ with an embedded representative and given as a reduced word in the generators m_i, a_j, b_j as in (102). Corollary 5.2 then implies that the Poisson bracket of a Wilson loop observable f_λ associated with a conjugation-invariant function $f \in \mathcal{C}^\infty(H)$ with functions $g_y, g \in \mathcal{C}^\infty(H)$, $y \in \{m_1, \ldots, b_g\}$, of the holonomies along the generators is given by

$$\begin{aligned}
\{g_y, f_\lambda\} &= t^{ab} R_a g_y \left(\sum_{k: t_k s_{k+1} \cap y = s_y} \epsilon(y, t_k s_{k+1}) R_b f_{\lambda_k} \right) \\
&\quad - t^{ab} L_a g_y \left(\sum_{k: t_k s_{k+1} \cap y = t_y} \epsilon(y, t_k s_{k+1}) R_b f_{\lambda_k} \right) \\
&= \sum_{k: t_k s_{k+1} \cap y = s_y} \epsilon(y, t_k s_{k+1}) \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g \circ \left(\rho_y \cdot G^t_{f,\lambda_k}\right) \\
&\quad + \sum_{k: t_k s_{k+1} \cap y = t_y} \epsilon(y, t_k s_{k+1}) \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g \circ \left(G^t_{f,\lambda_k} \cdot \rho_y\right),
\end{aligned} \tag{109}$$

where λ_k are the cyclic permutations of the expression (102) of λ as a reduced word in m_i, a_j, b_j:

$$\lambda_k := \left(z_k^{\delta_k} \cdots z_1^{\delta_1}\right) \lambda \left(z_k^{\delta_k} \cdots z_1^{\delta_1}\right)^{-1} = z_k^{\delta_k} \cdots z_1^{\delta_1} z_r^{\delta_r} \cdots z_{k+1}^{\delta_{k+1}} \qquad \forall k = 0, \ldots, t. \tag{110}$$

This suggests that the flow generated by the Wilson loop observable f_λ acts on the holonomies along the generators m_i, a_j, b_j by right multiplication with the functions G^t_{f,λ_k} associated with all segments $t_k s_{k+1}$ which intersect the generators at its starting point and by left multiplication with the functions G^t_{f,λ_k} for segments which intersect its endpoint. It turns out that this is the case and that the ordering of these factors G^t_{f,λ_k} is given by the order of the associated intersection points on the generator.

Theorem 5.4. *Let $\lambda \in \pi_1(S_{g,n} \setminus D)$ be an element with an embedded representative, given as a reduced word in the generators m_i, a_j, b_j as in (101) and with associated cyclic permutations (109). Represent λ graphically as described in section 3.2 and consider its intersection points with a generator $y \in \{m_1, \ldots, b_g\}$. Denote by $t_{i_1} s_{i_1+1}, \ldots, t_{i_k} s_{i_k+1}$ the segments of λ which intersect y above its starting point such that $t_{i_n} s_{i_n+1}$ lies below $t_{i_m} s_{i_m+1}$ for $n < m$ and by $t_{j_1} s_{j_1+1}, \ldots, t_{j_l} s_{j_l+1}$ the segments which intersect y above its endpoint such that $t_{j_n} s_{j_n+1}$ lies below $t_{j_m} s_{j_m+1}$ for $n < m$. Let $\epsilon_{i_n} = \epsilon(y, t_{i_n} s_{i_n+1})$, $\epsilon_{j_n} = \epsilon(y, t_{j_n} s_{j_n+1})$ the associated oriented intersection numbers. Then, the one-parameter group of diffeomorphisms*

$$\begin{aligned}
&T^t_{f,\lambda} : H^{n+2g} \to H^{n+2g} \\
&\frac{\mathrm{d}}{\mathrm{d}t} g \circ T^t_{f,\lambda}|_{t=0} = \{g, f_\lambda\} \qquad \forall g \in \mathcal{C}^\infty(H^{n+2g})
\end{aligned} \tag{111}$$

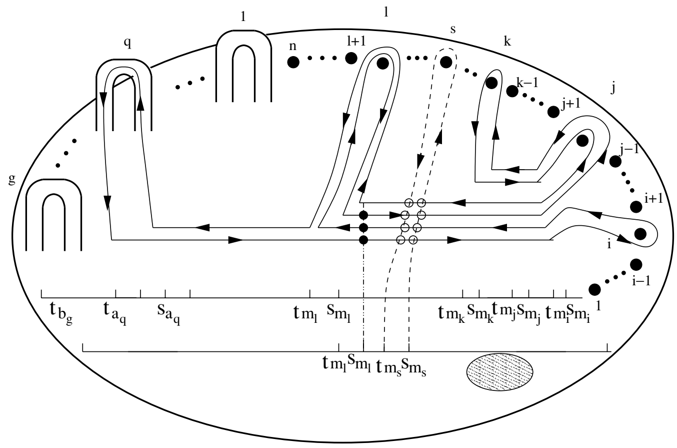

Figure 10. The graphical representation of $\lambda = a_q \circ m_l \circ m_j^{-1} \circ m_k \circ m_j \circ m_l^{-1} \circ m_i$ (full line), its intersection points with the generators m_s (dashed line, white circles) and with a vertical line based at s_{m_l} (black circles).

generated by the Wilson loop observable f_λ acts on the group element representing the holonomy along y according to

$$T^t_{f,\lambda} : Y \mapsto G^{t\epsilon_{j_1}}_{f,\lambda_{j_1}} G^{t\epsilon_{j_2}}_{f,\lambda_{j_2}} \cdots G^{t\epsilon_{j_l}}_{f,\lambda_{j_l}}(M_1, \ldots, B_g) \cdot Y \cdot G^{t\epsilon_{i_k}}_{f,\lambda_{i_k}} \cdots G^{t\epsilon_{i_2}}_{f,\lambda_{i_2}} G^{t\epsilon_{i_1}}_{f,\lambda_{i_1}}(M_1, \ldots, B_g), \tag{112}$$

where $G^t_{f,\lambda} : H^{n+2g} \to H$ is given by (107) and λ_k by (110).

Proof. To prove the theorem, we need to show that the derivatives at $t = 0$ of the functions in (112) agree with the Poisson brackets (101) and that the expression (112) defines a one-parameter group of diffeomorphism. The first statement follows directly from formula (109). To show that the maps $T^t_{f,\lambda} : H^{n+2g} \to H^{n+2g}$ define a one-parameter group of diffeomorphisms, we have to determine how the group elements associated with the curves λ_{i_n} and λ_{j_n} transform under $T^t_{f,\lambda}$. For this we note that these curves can be obtained by conjugating the curve representing λ with a vertical segment from the line containing the starting and endpoints to the segment $t_{z_{i_n}} s_{z_{i_n+1}}, t_{z_{j_n}} s_{z_{j_n+1}}$ as shown in figure 10. This vertical segment intersects the horizontal segments $t_{z_{i_m}} s_{z_{i_m}+1}, t_{z_{j_m}} s_{z_{j_m}+1}$ with $m < n$. Hence, we find that the transformation of the holonomies along $\lambda_{i_n}, \lambda_{j_n}$ is given by

$$\begin{aligned}
\rho_{\lambda_{i_n}} \circ T^t_{f,\lambda} &= G^{-t\epsilon_{i_1}}_{f,\lambda_{i_1}} G^{-t\epsilon_{i_2}}_{f,\lambda_{i_2}} \cdots G^{-t\epsilon_{i_{n-1}}}_{f,\lambda_{i_{n-l}}} \cdot \rho_{\lambda_{jn}} \cdot G^{t\epsilon_{i_{n-1}}}_{f,\lambda_{i_{n-l}}} \cdots G^{t\epsilon_{i_2}}_{f,\lambda_{i_2}} G^{t\epsilon_{i_1}}_{f,\lambda_{i_1}} \qquad n = 0, \ldots, k \\
\rho_{\lambda_{j_n}} \circ T^t_{f,\lambda} &= G^{t\epsilon_{j_1}}_{f,\lambda_{j_1}} G^{t\epsilon_{j_2}}_{f,\lambda_{j_2}} \cdots G^{t\epsilon_{j_{n-1}}}_{f,\lambda_{j_{n-l}}} \cdot \rho_{\lambda_{jn}} \cdot G^{-t\epsilon_{j_{n-1}}}_{f,\lambda_{j_{n-l}}} \cdots G^{-t\epsilon_{j_2}}_{f,\lambda_{j_2}} G^{-t\epsilon_{j_1}}_{f,\lambda_{j_1}} \qquad n = 0, \ldots, l,
\end{aligned} \tag{113}$$

which implies

$$\begin{aligned}
G^s_{f,\lambda_{i_n}} \circ T^t_{f,\lambda} &= G^{-t\epsilon_{i_1}}_{f,\lambda_{i_1}} G^{-t\epsilon_{i_2}}_{f,\lambda_{i_2}} \cdots G^{-t\epsilon_{i_{n-1}}}_{f,\lambda_{i_{n-l}}} \cdot G^s_{f,\lambda_{i_n}} \cdot G^{t\epsilon_{i_{n-1}}}_{f,\lambda_{i_{n-l}}} \cdots G^{t\epsilon_{i_2}}_{f,\lambda_{i_2}} G^{t\epsilon_{i_1}}_{f,\lambda_{i_1}} \qquad n = 0, \ldots, k \\
G^s_{f,\lambda_{j_n}} \circ T^t_{f,\lambda} &= G^{t\epsilon_{j_1}}_{f,\lambda_{j_1}} G^{t\epsilon_{j_2}}_{f,\lambda_{j_2}} \cdots G^{t\epsilon_{j_{n-1}}}_{f,\lambda_{j_{n-l}}} \cdot G^s_{f,\lambda_{j_n}} \cdot G^{-t\epsilon_{j_{n-1}}}_{f,\lambda_{j_{n-l}}} \cdots G^{-t\epsilon_{j_2}}_{f,\lambda_{j_2}} G^{-t\epsilon_{j_1}}_{f,\lambda_{j_1}} \qquad n = 0, \ldots, l.
\end{aligned} \tag{114}$$

By inserting (114) into the definition (112) of $T^t_{f,\lambda}$ we obtain after some calculation

$$T^s_{f,\lambda} \circ T^t_{f,\lambda} = T^{s+t}_{f,\lambda} \qquad \forall t, s \in \mathbb{R}. \tag{115}$$

□

Hence, we find that the one-parameter group of diffeomorphisms of H^{n+2g} generated by a generalized Wilson loop observable f_λ acts on the holonomies along the generators of the fundamental group by left and right multiplication with the functions G^t_{f,λ_k} associated with the segments $t_k s_{k+1}$ which intersect the corresponding curve on the surface. The ordering of the different factors G^t_{f,λ_k} is given by the vertical ordering of these segments, which agrees with the order in which the intersection points occur on the generator for intersection points above its starting point and is the opposite for intersection points above its endpoint. The unique vertical ordering of the segments $t_k s_{k+1}$ in the graphical representation of λ is therefore crucial to ensure that expression (112) defines a one-parameter group of diffeomorphisms. This unique ordering of the segments is a direct consequence of the fact that λ can be represented by an embedded curve. While formulae (98), (101), (103) for the Poisson brackets are valid for general elements $\lambda \in \pi_1(S_{g,n}\backslash D)$, the associated one-parameter groups of diffeomorphisms on H^{n+2g} generated by the observables f_λ via the Poisson bracket are not of the form (112) for elements without embedded representatives.

The fact that the transformations $T^t_{f,\lambda} : H^{n+2g} \to H^{n+2g}$ are generated via the Poisson bracket allows one to immediately deduce some of their properties. We have the following corollary, see also the discussion in [11].

Corollary 5.5. *Consider $\lambda, \eta \in \pi_1(S_{g,n}\backslash D)$ with embedded representatives and general conjugation-invariant functions $f, h \in \mathcal{C}^\infty(H)$. Then, the associated one-parameter groups of diffeomorphisms $T^t_{f,\lambda}, T^t_{h,\eta} : H^{n+2g} \to H^{n+2g}$ given by (112) have the following properties:*

(1) $T^t_{f,\lambda}$ acts by Poisson isomorphisms

$$\left\{g \circ T^t_{f,\lambda}, k \circ T^t_{f,\lambda}\right\} = \{g, k\} \circ T^t_{f,\lambda} \qquad \forall g, k \in \mathcal{C}^\infty(H^{n+2g}). \tag{116}$$

(2) $T^t_{f,\lambda}$ leaves the constraint (62) invariant, commutes with simultaneous conjugation of all arguments of H^{n+2g} by H and acts on the first n components by conjugation.

(3) The transformations $T^t_{f,\lambda}, T^t_{h,\eta}$ commute if and only if

$$\langle g_f(u), g_h(u)\rangle = 0 \qquad \forall u \in H^{n+2g} \tag{117}$$

or η and λ are conjugated to elements with representatives that do not intersect.

Proof. Identity (116) is a direct consequence of the fact that the transformations $T^t_{f,\lambda}$ are generated by Hamiltonians. That it leaves the constraint invariant and commutes with the associated gauge transformations follows from the fact that it is generated by a gauge-invariant function which Poisson commutes with functions of the constraint (62). That it acts on the first n components of H^{n+2g} by conjugation follows directly from formula (98), (101) for the Poisson bracket. Finally, we note that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}s\,\mathrm{d}t} h \circ T^t_{f,\lambda} \circ T^s_{g,\eta}|_{t=s=0} - \frac{\mathrm{d}^2}{\mathrm{d}s\,\mathrm{d}t} h \circ T^s_{g,\eta} \circ T^t_{f,\lambda}|_{t=s=0} \\ = \{\{h, f_\lambda\}, g_\eta\} - \{\{h, g_\eta\}, f_\lambda\} = \{h, \{f_\lambda, g_\eta\}\}.\end{aligned}$$

From expression (103) for the Poisson bracket and taking into account the identity $t^{ab} R_a f(u) R_b h(u) = \langle g_f(u), g_h(u)\rangle \forall u \in H$ one then obtains (117). □

5.3. The flows generated by the Wilson loop observables: the algebraic formulation

Formula (112) gives an explicit expression for the transformation generated by the generalized Wilson loop observables associated with elements $\lambda \in \pi_1(S_{g,n}\setminus D)$ with embedded representatives. However, it still relies on a graphical procedure to determine the order in which the intersection points of λ occur on the representatives of each generator m_i, a_j, b_j. We will now use the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$ to derive a purely algebraic formulation, in which the flows $T^t_{f,\lambda}$ are characterized entirely in terms of the expression of λ as a reduced word in the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$. For this, we recall the discussion from section 3.1 where we demonstrated how the order in which the intersection points occur on each generator m_i, a_j, b_j can be derived from the expression of λ as a cyclically reduced word in the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$:

$$\lambda = \overline{x}_r^{\alpha_r} \cdots \overline{x}_1^{\alpha_1} \qquad \overline{x}_r^{\alpha_r} \neq \overline{x}_1^{-\alpha_1}. \tag{118}$$

It is shown there that the intersection points of λ with generators $x \in \{a_1, \ldots, b_g\}$ are in one-to-one correspondence with cyclic permutations $\tau \in \mathrm{CPerm}(\lambda)$ with $\mathrm{LF}(\tau) = \overline{x}$ and that each element $\tau \in \mathrm{CPerm}(\lambda)$ with $\mathrm{LF}(\tau) = \overline{m}_i$ corresponds to a pair of intersection points of λ with m_i, one at its starting point and one at its endpoint and with opposite intersection numbers. For a given factor $\overline{x}_k^{\alpha_k}$ in (118), the associated element $\tau \in \mathrm{CPerm}(\lambda)$ with $\mathrm{LF}(\tau) = \overline{x}_k$ is given by

$$\tau = \begin{cases} \overline{x}_{k-1}^{\alpha_{k-1}} \cdots \overline{x}_1^{\alpha_1} \overline{x}_r^{\alpha_r} \cdots \overline{x}_k^{\alpha_k} & \alpha_k = 1 \\ \left(\overline{x}_k^{\alpha_k} \cdots \overline{x}_1^{\alpha_1} \overline{x}_r^{\alpha_r} \cdots \overline{x}_{k+1}^{\alpha_{k+1}}\right)^{-1} & \alpha_k = -1. \end{cases} \tag{119}$$

The elements $\lambda_{i_n}, \lambda_{j_n}$ in (112) are obtained from the cyclic permutations τ via the assignment of intersection points between the different factors in the expression of λ as a reduced word in the generators m_i, a_j, b_j given in theorem 3.2. Using formulae (30), (31), (33), (34) for the splitting of the dual generators and setting $\mathrm{sgn}(\tau) = 1$ if τ is a cyclic permutation of λ as a cyclically reduced word in $\overline{m}_i, \overline{a}_j, \overline{b}_j$ and $\mathrm{sgn}(\tau) = -1$ if it is a cyclic permutation of λ^{-1}, we find that the cyclic permutations associated with the segments λ_{i_n} which intersect the generators at their starting points are given by

$$\lambda_{i_n} = \begin{cases} (m_{i-1} \cdots m_1)\tau_{i_n}^{\mathrm{sgn}(\tau_{i_n})}(m_{i-1} \cdots m_1)^{-1} & \text{for} \quad \mathrm{LF}(\tau_{i_n}) = \overline{m}_i \\ (h_{j-1} \cdots m_1)\tau_{i_n}^{\mathrm{sgn}(\tau_{i_n})}(h_{j-1} \cdots m_1)^{-1} & \text{for} \quad \mathrm{LF}(\tau_{i_n}) = \overline{a}_j \\ \left(b_j^{-1} a_j h_{j-1} \cdots m_1\right)\tau_{i_n}^{\mathrm{sgn}(\tau_{i_n})}\left(b_j^{-1} a_j h_{j-1} \cdots m_1\right)^{-1} & \text{for} \quad \mathrm{LF}(\tau_{i_n}) = \overline{b}_j. \end{cases} \tag{120}$$

Similarly, the cyclic permutations for segments λ_{j_n} which intersect the generators at their endpoints take the form

$$\lambda_{j_n} = \begin{cases} (m_{i-1} \cdots m_1)\tau_{j_n}^{\mathrm{sgn}(\tau_{j_n})}(m_{i-1} \cdots m_1)^{-1} & \text{for} \quad \mathrm{LF}(\tau_{j_n}) = \overline{m}_i \\ (a_j h_{j-1} \cdots m_1)\tau_{j_n}^{\mathrm{sgn}(\tau_{j_n})}(a_j h_{j-1} \cdots m_1)^{-1} & \text{for} \quad \mathrm{LF}(\tau_{j_n}) = \in \{\overline{a}_j, \overline{b}_j\}. \end{cases} \tag{121}$$

Furthermore, we note that the oriented intersection numbers $\varepsilon_{i_n}, \varepsilon_{j_n}$ in (112) are given by $\mathrm{sgn}(\tau)$ for $\mathrm{LF}(\tau) = \overline{a}_j$, by $-\mathrm{sgn}(\tau)$ for $\mathrm{LF}(\tau) = \overline{b}_j$. For $\mathrm{LF}(\tau) = \overline{m}_i$, $\mathrm{sgn}(\tau)$ gives the oriented intersection number of the intersection point at the starting point of m_i, which is the opposite of the one for its endpoint.

Hence, if we denote by $\tau_{i_n}, \tau_{j_n} \in \mathrm{CPerm}(\lambda)$ the cyclic permutations associated via (119), respectively, with elements $\lambda_{i_n} \in \pi_1(S_{g,n}\setminus D)$ at the right of Y and elements $\lambda_{j_n} \in \pi_1(S_{g,n}\setminus D)$ in (112), we find using (108)

$$G^{t\varepsilon_{i_n}}_{f,\lambda_{i_n}} = \begin{cases} (M_{i-1}\cdots M_1)G^{t\,\mathrm{sgn}(\tau_{i_n})}_{f,|\tau_{i_n}|}(M_{i-1}\cdots M_1)^{-1} & \text{for} \quad \mathrm{LF}(\tau_{i_n}) = \overline{m}_i \\ (H_{j-1}\cdots M_1)G^{t\,\mathrm{sgn}(\tau_{i_n})}_{f,|\tau_{i_n}|}(H_{j-1}\cdots M_1)^{-1} & \text{for} \quad \mathrm{LF}(\tau_{i_n}) = \overline{a}_j \\ \left(B_j^{-1}A_jH_{j-1}\cdots M_1\right)G^{-t\,\mathrm{sgn}(\tau_{i_n})}_{f,|\tau_{i_n}|}\left(B_j^{-1}A_jH_{j-1}\cdots M_1\right)^{-1} & \text{for} \quad \mathrm{LF}(\tau_{i_n}) = \overline{b}_j \end{cases} \tag{122}$$

$$G^{t\varepsilon_{j_n}}_{f,\lambda_{j_n}} = \begin{cases} (M_{i-1}\cdots M_1)G^{-t\,\mathrm{sgn}(\tau_{i_n})}_{f,|\tau_{i_n}|}(M_{i-1}\cdots M_1)^{-1} & \text{for} \quad \mathrm{LF}(\tau_{j_n}) = \overline{m}_i \\ (A_jH_{j-1}\cdots M_1)G^{t\,\mathrm{sgn}(\tau_{i_n})}_{f,|\tau_{j_n}|}(A_jH_{j-1}\cdots M_1)^{-1} & \text{for} \quad \mathrm{LF}(\tau_{j_n}) = \overline{a}_j \\ (A_jH_{j-1}\cdots M_1)G^{-t\,\mathrm{sgn}(\tau_{i_n})}_{f,|\tau_{j_n}|}(A_jH_{j-1}\cdots M_1)^{-1} & \text{for} \quad \mathrm{LF}(\tau_{j_n}) = \overline{b}_j \end{cases}$$

where $|\tau| = \tau^{\mathrm{sgn}(\tau)}$ for $\tau \in \mathrm{CPerm}(\lambda)$. We now move all factors $G^{t\varepsilon_{j_n}}_{f,\lambda_{j_n}}$ in (112) to the right of $Y = A_j$ by conjugating them with A_j^{-1} and all factors $G^{t\varepsilon_{i_n}}_{f,\lambda_{i_n}}$ to the left of $Y = B_j$ by conjugating them with B_j^{-1}. Figure 4 implies that the order of the intersection points on the generator a_j is the order of the associated intersection points on the side a_j of the polygon $P^D_{g,n}$ and that the order of the intersection points on b_j is the opposite of the order of intersection points on the side b_j of $P^D_{g,n}$. In the case of the generators m_i, the order of the intersection points at the starting point of m_i agrees with the order of the corresponding points on $P^D_{g,n}$, while the order of the intersection points at its endpoint is the opposite.

Theorem 5.6. *For any embedded $\lambda \in \pi_1(S_{g,n}\backslash D)$ given as a cyclically reduced words in the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$ and any conjugation invariant $f \in \mathcal{C}^\infty(H)$, the transformation $T^t_{f,\lambda}$ generated by the observable f_λ is given by*

$$\begin{aligned} T^t_{f,\lambda} : M_i \mapsto\ & (M_{i-1}\cdots M_1)\left(\prod^{\rightarrow}_{\substack{\tau\in CPerm(\lambda)\\ \mathrm{LF}(\tau)=\overline{m}_i}} G^{t\,sgn(\tau)}_{f,|\tau|}\right)^{-1}(M_{i-1}\cdots M_1)^{-1}M_i(M_{i-1}\cdots M_1) \\ & \times\left(\prod^{\rightarrow}_{\substack{\tau\in CPerm(\lambda)\\ \mathrm{LF}(\tau)=\overline{m}_i}} G^{t\,sgn(\tau)}_{f,|\tau|}\right)(M_{i-1}\cdots M_1)^{-1} \\ A_j \mapsto\ & A_j\cdot(H_{j-1}\cdots M_1)\left(\prod^{\rightarrow}_{\substack{\tau\in CPerm(\lambda)\\ \mathrm{LF}(\tau)=\overline{a}_j}} G^{t\,sgn(\tau)}_{f,|\tau|}\right)(H_{j-1}\cdots M_1)^{-1} \\ B_j \mapsto\ & (A_jH_{j-1}\cdots M_1)\left(\prod^{\rightarrow}_{\substack{\tau\in CPerm(\lambda)\\ \mathrm{LF}(\tau)=\overline{b}_j}} G^{t\,sgn(\tau)}_{f,|\tau|}\right)^{-1}(A_jH_{j-1}\cdots M_1)^{-1}\cdot B_j, \end{aligned} \tag{123}$$

where the ordering of the factors is the one defined by (27), (28). More precisely, for $\tau, \eta \in CPerm(\lambda)$, $\mathrm{LF}(\tau) = \mathrm{LF}(\eta) = \overline{x}$, we have

$$\tau < \eta \quad \Leftrightarrow \quad \mathrm{LF}(\tau_s) = \mathrm{LF}(\eta_s)\forall 1 \leqslant s \leqslant k, \mathrm{LF}\left(\tau_k^{-1}\right) >_{\mathrm{LF}(\tau_k)} \mathrm{LF}\left(\eta_k^{-1}\right) \tag{124}$$

with the ordering $>_x$ defined as in (24) and τ_k, η_k denoting the cyclic permutations (17). If $\{\tau \in CPerm(\lambda)|\mathrm{LF}(\tau) = \overline{x}\} = \{\tau_1, \ldots, \tau_s\}$ and $\tau_1 < \cdots < \tau_s$ with respect to the ordering

(124), then the associated product is given by

$$\overrightarrow{\prod_{\substack{\tau\in CPerm(\lambda)\\ \mathrm{LF}(\tau)=\overline{x}}}} G^t_{f,\tau} = G^t_{f,\tau_s}\cdot G^t_{f,s-1}\cdots G^t_{f,\tau_1}. \tag{125}$$

By applying the involution $I \in \mathrm{Aut}(\pi_1(S_{g,n}\setminus D))$ to Fock and Rosly's description [3] of the moduli space of flat connections, we therefore obtain explicit expressions for the Poisson brackets of generalized Wilson loop observables associated with curves on $S_{g,n}\setminus D$ and the associated flows on phase space. These expressions generalize the results of Goldman, see in particular theorem 3.5. and formulae (4.4), (4.6) in [11], to the case of surfaces with punctures. While the results in [11] are obtained via cohomological methods and characterize flows and Poisson brackets geometrically in terms of the intersection behaviour of curves on the surfaces, theorems 5.1, 5.4 and 5.6 give concrete expressions in terms of the holonomies along the generators of the fundamental group $\pi_1(S_{g,n}\setminus D)$. Moreover, the resulting formulae are purely algebraic and characterize flows and Poisson brackets in terms of the expression of the associated curves as reduced words in the dual generators.

This explicitness presents an advantage in physical applications of the theory such as the Chern–Simons formulation of (2+1)-dimensional gravity and, more generally, the quantization of Chern–Simons theory. Most approaches to quantization such as Alekseev, Grosse and Schomerus' combinatorial quantization formalism [6–8] for Chern–Simons theory with compact semisimple gauge groups and the quantization procedures in [9] and [10, 15] for, respectively, gauge group $SL(2,\mathbb{C})$ and semidirect product gauge groups $G\ltimes\mathfrak{g}^*$ are based on Fock and Rosly's description of the moduli space and take the holonomies along a set of generators of the fundamental group as their basic variables. The formulation in this paper could therefore serve as a framework for the investigation of the associated observables and transformations in quantized Chern–Simons theory. Moreover, it is used in [12] to investigate the role of these flows in the Chern–Simons formulation of (2+1)-dimensional gravity and to relate them to the geometrical construction of evolving (2+1)-spacetimes.

5.4. Example

We conclude this section by determining the transformations $T^t_{f,\lambda}$ for a concrete example. We consider the element

$$\begin{aligned}\lambda &= a_q\circ m_l\circ m_j^{-1}\circ m_k\circ m_j\circ m_l^{-1}\circ m_i \qquad 1\leqslant i<j<k<l\leqslant n,\ q\in\{1,\dots,g\}\\ &= \left(\overline{m}_1^{-1}\cdots\overline{m}_{i-1}^{-1}\right)\circ\left(\overline{m}_i^{-1}\cdots\overline{h}_{q-1}^{-1}\overline{a}_q^{-1}\overline{b}_q\overline{a}_q\right)\circ\left(\overline{h}_{q-1}\cdots\overline{m}_{l+1}\right)\circ\left(\overline{m}_{l-1}\cdots\overline{m}_j\right)\\ &\quad\circ\left(\overline{m}_{j+1}^{-1}\cdots\overline{m}_k^{-1}\right)\circ\left(\overline{m}_{k-1}\cdots\overline{m}_{j+1}\right)\circ\left(\overline{m}_j^{-1}\cdots\overline{m}_{l-1}^{-1}\right)\circ\left(\overline{m}_l\cdots\overline{m}_{i+1}\right)\circ\left(\overline{m}_{i-1}\cdots\overline{m}_1\right),\end{aligned} \tag{126}$$

whose graphical representation is shown in figure 10. By using the graphical representation in figure 10 and the definitions (112), (110) we can determine the action of the one-parameter group of diffeomorphisms $T^t_{f,\lambda}$ associated with a conjugation-invariant function $f\in\mathcal{C}^\infty(H^{n+2g})$. We find that their action on the holonomies along the generators m_i, a_j, b_j is given by

$$\begin{aligned}&M_i \mapsto G^{-t}_{f,\lambda}\cdot M_i\cdot G^t_{f,\lambda}\\ &M_s \mapsto G^{-t}_{f,\lambda}G^t_{f,\lambda_1}\cdot M_s\cdot G^{-t}_{f,\lambda_1}G^t_{f,\lambda} \qquad \forall i<s<j\\ &M_j \mapsto G^{-t}_{f,\lambda}G^t_{f,\lambda_1}G^{-t}_{f,\lambda_2}G^t_{f,\lambda_3}\cdot M_j\cdot G^{-t}_{f,\lambda_3}G^t_{f,\lambda_2}G^{-t}_{f,\lambda_1}G^t_{f,\lambda}\\ &M_s \mapsto G^{-t}_{f,\lambda}G^t_{f,\lambda_1}G^{-t}_{f,\lambda_2}G^t_{f,\lambda_3}G^{-t}_{f,\lambda_4}G^t_{f,\lambda_5}\cdot M_s\cdot G^{-t}_{f,\lambda_5}G^t_{f,\lambda_4}G^{-t}_{f,\lambda_3}G^t_{f,\lambda_2}G^{-t}_{f,\lambda_1}G^t_{f,\lambda} \qquad j<s<k\end{aligned}$$

$$\begin{aligned}
&M_k \mapsto G^{-t}_{f,\lambda}G^t_{f,\lambda_1}G^{-t}_{f,\lambda_2}G^t_{f,\lambda_3}G^{-t}_{f,\lambda_4}\cdot M_k\cdot G^t_{f,\lambda_4}G^{-t}_{f,\lambda_3}G^t_{f,\lambda_2}G^{-t}_{f,\lambda_1}G^t_{f,\lambda}\\
&M_s \mapsto G^{-t}_{f,\lambda}G^t_{f,\lambda_1}G^{-t}_{f,\lambda_2}G^t_{f,\lambda_3}\cdot M_s\cdot G^{-t}_{f,\lambda_3}G^t_{f,\lambda_2}G^{-t}_{f,\lambda_1}G^t_{f,\lambda} \qquad k<s<l\\
&M_l \mapsto G^{-t}_{f,\lambda}G^t_{f,\lambda_1}\cdot M_l\cdot G^{-t}_{f,\lambda_1}G^t_{f,\lambda}\\
&Y \mapsto G^{-t}_{f,\lambda}G^t_{f,\lambda_6}\cdot Y\cdot G^{-t}_{f,\lambda_6}G^t_{f,\lambda} \qquad Y\in\{M_{l+1},\ldots,M_n,A_1,B_1,\ldots,A_{q-1},B_{q-1}\}\\
&A_q \mapsto G^{-t}_{f,\lambda}\cdot A_q\cdot G^t_{f,\lambda}\\
&B_q \mapsto B_q\cdot G^t_{f,\lambda},
\end{aligned}\tag{127}$$

where we omit the argument $(M_1,\ldots,B_g)$ of the functions $G^t_{f,\eta}$, list only those generators which do not transform trivially and set

$$\begin{aligned}
&\lambda_0=\lambda\\
&\lambda_1=m_i\circ\lambda\circ m_i^{-1}=m_i\circ a_q\circ m_l\circ m_j^{-1}\circ m_k\circ m_j\circ m_l^{-1}\\
&\lambda_2=\left(m_l^{-1}m_i\right)\circ\lambda\circ\left(m_l^{-1}m_i\right)^{-1}=m_l^{-1}\circ m_i\circ a_q\circ m_l\circ m_j^{-1}\circ m_k\circ m_j\\
&\lambda_3=(a_qm_l)^{-1}\circ\lambda\circ(a_qm_l)=m_j^{-1}\circ m_k\circ m_j\circ m_l^{-1}\circ a_q\circ m_l\\
&\lambda_4=\left(a_qm_lm_j^{-1}\right)^{-1}\circ\lambda\circ\left(a_qm_lm_j^{-1}\right)=m_k\circ m_j\circ m_l^{-1}\circ a_q\circ m_l\circ m_j^{-1}\\
&\lambda_5=\left(m_jm_l^{-1}m_i\right)\circ\lambda\circ\left(m_jm_l^{-1}m_i\right)^{-1}=m_j\circ m_l^{-1}\circ m_i\circ a_q\circ m_l\circ m_j^{-1}\circ m_k\\
&\lambda_6=a_q^{-1}\circ\lambda\circ a_q=m_l\circ m_j^{-1}\circ m_k\circ m_j\circ m_l^{-1}\circ a_q.
\end{aligned}\tag{128}$$

The ordering of the horizontal segments in figure 10 then implies that the transformation of the holonomies along the cyclic permutations λ_i is given by

$$\begin{aligned}
&\rho_\lambda\circ T^t_{f,\lambda}=\rho_\lambda\\
&\rho_{\lambda_1}\circ T^t_{f,\lambda}=G^{-t}_{f,\lambda}\cdot\rho_{\lambda_1}\cdot G^{-t}_{f,\lambda}\\
&\rho_{\lambda_2}\circ T^t_{f,\lambda}=G^{-t}_{f,\lambda}G^t_{f,\lambda_1}\cdot\rho_{\lambda_2}\cdot G^t_{f,\lambda_1}G^{-t}_{f,\lambda}\\
&\rho_{\lambda_3}\circ T^t_{f,\lambda}=G^{-t}_{f,\lambda}G^t_{f,\lambda_1}G^t_{f,\lambda_2}\cdot\rho_{\lambda_3}\cdot G^{-t}_{f,\lambda_2}G^t_{f,\lambda_1}G^{-t}_{f,\lambda}\\
&\rho_{\lambda_4}\circ T^t_{f,\lambda}=G^{-t}_{f,\lambda}G^t_{f,\lambda_1}G^t_{f,\lambda_2}G^{-t}_{f,\lambda_3}\cdot\rho_{\lambda_4}\cdot G^t_{f,\lambda_3}G^{-t}_{f,\lambda_2}G^t_{f,\lambda_1}G^{-t}_{f,\lambda}\\
&\rho_{\lambda_5}\circ T^t_{f,\lambda}=G^{-t}_{f,\lambda}G^t_{f,\lambda_1}G^t_{f,\lambda_2}G^{-t}_{f,\lambda_3}G^t_{f,\lambda_4}\cdot\rho_{\lambda_5}\cdot G^{-t}_{f,\lambda_4}G^t_{f,\lambda_3}G^{-t}_{f,\lambda_2}G^t_{f,\lambda_1}G^{-t}_{f,\lambda}\\
&\rho_{\lambda_6}\circ T^t_{f,\lambda}=G^{-t}_{f,\lambda}\cdot\rho_{\lambda_6}\cdot G^{-t}_{f,\lambda}.
\end{aligned}\tag{129}$$

A straightforward but lengthy calculation shows that this agrees with the transformation obtained from the expressions (128) and the transformation (127) for the holonomies along the generators.

6. Dual generators and the mapping class group

In this section, we apply the results of section 5 to a specific set of generalized Wilson loop observables which are constructed from the Ad-invariant symmetric bilinear form $\langle,\rangle$ in the Chern–Simons action. We show that the associated one-parameter groups of diffeomorphisms are related to the Dehn twists around embedded curves on the surface $S_{g,n}\setminus D$. This allows us to determine the transformation of Fock and Rosly's Poisson structure under the action of the mapping class group $\mathrm{Map}(S_{g,n}\setminus D)$ and under general automorphisms of $\pi_1(S_{g,n}\setminus D)$ which arise from homeomorphisms of $S_{g,n}\setminus D$.

The mapping class group $\mathrm{Map}(S_{g,n}\setminus D)$ is the group of equivalence classes of orientation-preserving homeomorphisms of $S_{g,n}\setminus D$ which fix the punctures as a set and fix the boundary of the disc D pointwise. Two homeomorphisms are identified if they differ by one which is isotopic to the identity. The pure mapping class group $\mathrm{PMap}(S_{g,n}\setminus D)$ is the subgroup of $\mathrm{Map}(S_{g,n}\setminus D)$ which contains the equivalence classes of homeomorphisms that fix each of the punctures and is related to the mapping class group $\mathrm{Map}(S_{g,n}\setminus D)$ by the short exact sequence

$$1 \to \mathrm{PMap}(S_{g,n}\setminus D) \xrightarrow{i} \mathrm{Map}(S_{g,n}\setminus D) \xrightarrow{\pi} S_n \to 1, \tag{130}$$

where i is the canonical embedding and $\pi : \mathrm{Map}(S_{g,n}\setminus D) \to S_n$ is the projection onto the symmetric group that assigns to each element of the mapping class group the associated permutation of the punctures. As explained in [16, 17], the pure mapping class group $\mathrm{PMap}(S_{g,n}\setminus D)$ is generated by Dehn twists around a set of embedded curves, and a set of generators of the full mapping class group $\mathrm{Map}(S_{g,n})$ is obtained by supplementing this set with $n-1$ elements which get mapped to the elementary transpositions via π. A set of generators of the pure and full mapping class group and their action on the fundamental group is given in the appendix. Each element of the mapping class group induces a unique automorphism of the fundamental group $\pi_1(S_{g,n}\setminus D)$ which maps the loop m_D around the disc to itself and acts on the loops around the punctures by conjugation and permutations. Via the identification of the generators m_i, a_j, b_j with the different copies of H in the product H^{n+2g} each element $\varphi \in \mathrm{Map}(S_{g,n}\setminus D)$ then induces a diffeomorphism $\Phi_\varphi : H^{n+2g} \to H^{n+2g}$.

We consider the generalized Wilson loop observables associated with an element $\lambda \in \pi_1(S_{g,n}\setminus D)$ with an embedded representative and with the bilinear form $\langle , \rangle$ in the Chern–Simons action. Using the parametrization via the exponential map and composing, we obtain a conjugation-invariant function $\tilde{t} : H \to \mathbb{R}$ by setting

$$\tilde{t}(\mathrm{e}^{p^a J_a}) := \tfrac{1}{2}\langle p^a J_a, p^a J_a\rangle. \tag{131}$$

As we require the exponential map to be locally but not globally bijective, the parametrization by elements of the Lie algebra $\mathfrak{h}$ is in general not unique. To obtain a unique parametrization, one has to restrict the Lie algebra elements $p^a J_a$ appropriately, which implies that the function $\tilde{t}$ defined in (131) is locally but not globally $\mathcal{C}^\infty$. However, as we are only interested in the local properties of this map, we will not address this issue further. To determine the flows generated by the associated Wilson loop observables $\tilde{t}_\lambda$, we need to derive the maps $g_{\tilde{t}} : H \to \mathfrak{h}$, $G_{\tilde{t}} : H \to H$ defined in (104). We use a result by Goldman [11] summarized in the following lemma.

Lemma 6.1 (Goldman [11]). *Let $s \in \mathfrak{h}^* \otimes \mathfrak{h}^*$ be an Ad-invariant, symmetric bilinear form on $\mathfrak{h}$ and consider the associated function $\tilde{s} : H \to \mathbb{R}$, $\tilde{s}(\mathrm{e}^{p^a J_a}) = s(p^a J_a, p^a J_a)$. Then, the action of the left- and right-invariant vector fields on $\tilde{s}$ is given by*

$$R_a\tilde{s}(\mathrm{e}^{p^b J_b}) = -L_a\tilde{s}(\mathrm{e}^{p^b J_b}) = 2s(J_a, J_b)p^b. \tag{132}$$

By applying formula (132) to the function $\tilde{t} : H \to \mathbb{R}$, we find that the maps $g_{\tilde{t}} : H \to \mathfrak{h}$, $G_{\tilde{t}} : H \to H$ in (104) take the form

$$g_{\tilde{t}}(\mathrm{e}^{p^a J_a}) = p^a \qquad G^t_{\tilde{t}}(\mathrm{e}^{p^a J_a}) = \mathrm{e}^{t p^a J_a}. \tag{133}$$

For flow parameter $t = 1$, the diffeomorphism $G^t_{\tilde{t}} : H \to H$ is the identity map on H^{n+2g}. This implies that the associated flows $T^1_{\tilde{t},\lambda}$ defined by (112) act on the group elements associated with the generators of the fundamental group by left and right multiplication with the holonomies of certain elements in the conjugacy class of λ and correspond to an automorphism of the fundamental group $\pi_1(S_{g,n}\setminus D)$. Together with the dependence of this transformation on intersection points, this indicates that the transformation $T^1_{\tilde{t},\lambda}$ should be related to the diffeomorphism of H^{n+2g} induced by the Dehn twist around λ.

Theorem 6.2.

(1) For any embedded $\lambda \in \pi_1(S_{g,n}\setminus D)$*, the associated one-parameter group* $T^t_{\tilde{t},\lambda} : H^{n+2g} \to H^{n+2g}$ *of diffeomorphisms represents an infinitesimal Dehn twist around* λ

$$T^1_{\tilde{t},\lambda} = D_\lambda, \tag{134}$$

where $D_\lambda : H^{n+2g} \to H^{n+2g}$ *is the diffeomorphism of* H^{n+2g} *induced by the action* $d_\lambda \in Aut(\pi_1(S_{g,n}\setminus D))$ *of the Dehn twist around* λ *on the fundamental group* $\pi_1(S_{g,n}\setminus D)$.

(2) The mapping class group $Map(S_{g,n}\setminus D)$ *acts by Poisson isomorphisms*

$$\{f \circ \Phi_\varphi, g \circ \Phi_\varphi\} = \{f, g\} \circ \Phi_\varphi \qquad \forall f, g \in \mathcal{C}^\infty(H^{n+2g}), \quad \varphi \in Map(S_{g,n}\setminus D). \tag{135}$$

Proof. The general reasoning follows the proof in [15], where an analogous statement is proved for Chern–Simons theory with gauge groups of the form $H = G \ltimes \mathfrak{g}^*$.

1. We start by proving identity (134) for the set of generating Dehn twists given in the appendix. For this, we have to determine the Poisson brackets of the associated Wilson loop observables, which can be done either by expressing the curves in terms of the dual generators $\overline{m}_i, \overline{a}_j, \overline{b}_j$ and using (98) or via the graphical procedure in section 3.2 and formula (101). Expressed in both, the generators m_i, a_j, b_j and their duals, the curves (A.1) for the generating Dehn twists in the appendix are given by

$$\begin{aligned}
a_i &= \left(\overline{m}_1^{-1}\cdots\overline{h}_{i-1}^{-1}\right)\circ\overline{b}_i\circ\left(\overline{h}_{i-1}\cdots\overline{m}_1\right) \qquad i = 1,\dots,g\\
\delta_i &= a_i^{-1}\circ b_i^{-1}\circ a_i = \left(\overline{m}_1^{-1}\cdots\overline{h}_{i-1}^{-1}\right)\circ\overline{a}_i^{-1}\circ\left(\overline{h}_{i-1}\cdots\overline{m}_1\right) \qquad i = 1,\dots,g\\
\alpha_i &= a_i^{-1}\circ b_i^{-1}\circ a_i\circ b_{i-1} = \left(\overline{m}_1^{-1}\cdots\overline{h}_{i-1}^{-1}\right)\circ\overline{a}_i^{-1}\circ\overline{a}_{i-1}\circ\overline{h}_{i-1}^{-1}\circ\left(\overline{h}_{i-1}\cdots\overline{m}_1\right)\\
\epsilon_i &= a_i^{-1}\circ b_i^{-1}\circ a_i\circ h_{i-1}\cdots h_1 = \left(\overline{m}_1^{-1}\cdots\overline{m}_n\right)\circ\overline{h}_1^{-1}\cdots\overline{h}_{i-1}^{-1}\circ\overline{a}_i^{-1}\circ\left(\overline{m}_n\cdots\overline{m}_1\right)\\
&\qquad i = 2,\dots,g\\
\kappa_{\nu,\mu} &= m_\mu\circ m_\nu = \left(\overline{m}_1^{-1}\cdots\overline{m}_{\nu-1}^{-1}\right)\circ\left(\overline{m}_\nu^{-1}\cdots\overline{m}_\mu^{-1}\circ\overline{m}_{\mu-1}\cdots\overline{m}_{\nu+1}\right)\circ\left(\overline{m}_{\nu-1}\cdots\overline{m}_1\right)\\
&\qquad 1\leqslant\nu<\mu\leqslant n\\
\kappa_{\nu,n+2i-1} &= a_i^{-1}\circ b_i^{-1}\circ a_i\circ m_\nu = \left(\overline{m}_1^{-1}\cdots\overline{m}_{\nu-1}^{-1}\right)\circ\left(\overline{m}_\nu^{-1}\cdots\overline{h}_{i-1}^{-1}\circ\overline{a}_i^{-1}\circ\overline{h}_{i-1}\cdots\overline{m}_{\nu+1}\right)\\
&\qquad\circ\left(\overline{m}_{\nu-1}\cdots\overline{m}_1\right)\\
\kappa_{\nu,n+2i} &= b_i\circ m_\nu = \left(\overline{m}_1^{-1}\cdots\overline{m}_{\nu-1}^{-1}\right)\circ\left(\overline{m}_\nu^{-1}\cdots\overline{h}_i^{-1}\circ\overline{a}_i\circ\overline{h}_{i-1}\cdots\overline{m}_{\nu+1}\right)\circ\left(\overline{m}_{\nu-1}\cdots\overline{m}_1\right),
\end{aligned} \tag{136}$$

and a lengthy but direct calculation yields the Poisson brackets of the associated Wilson loop observables with a general function $f \in \mathcal{C}^\infty(H^{n+2g})$:

$$\begin{aligned}
\{f, \tilde{t}_{a_i}\} &= p^a_{a_i} R^{B_i}_a f\\
\{f, \tilde{t}_{\delta_i}\} &= p^a_{b_i} L^{A_i}_a f\\
\{f, \tilde{t}_{\alpha_i}\} &= p^a_{\alpha_i}\left(R^{A_i}_a + L^{A_{i-1}}_a + R^{B_{i-1}}_a + L^{B_{i-1}}_a\right) f\\
\{f, \tilde{t}_{\epsilon_i}\} &= p^a_{\epsilon_i}\left(R^{A_i}_a + \sum_{j=1}^{i-1} R^{A_j}_a + L^{A_j}_a + R^{B_j}_a + L^{B_j}_a\right) f\\
\{f, \tilde{t}_{\kappa_{\nu,\mu}}\} &= p^a_{\kappa_{\nu,\mu}}\left(R^{M_\nu}_a + L^{M_\nu}_a + R^{M_\mu}_a + L^{M_\mu}_a\right) f + \left(p^a_{\kappa_{\nu,\mu}} - p^a_{m_\nu\circ\kappa_{\nu,\mu}\circ m_\nu^{-1}}\right)\sum_{j=\nu+1}^{\mu-1}\left(R^{M_j}_a + L^{M_j}_a\right) f
\end{aligned}$$

$$
\begin{aligned}
\{f, \tilde{t}_{\kappa_{\nu,n+2i-1}}\} &= p^a_{\kappa_{\nu,n+2i-1}}\left(R_a^{M_\nu} + L_a^{M_\nu} + R_a^{A_i}\right) f + \left(p^a_{\kappa_{\nu,n+2i-1}} - p^a_{m_\nu \circ \kappa_{\nu,n+2i-1} \circ m_\nu^{-1}}\right) \\
&\quad \times \left(\sum_{j=\nu+1}^{n} R_a^{M_j} + L_a^{M_j} + \sum_{j=1}^{i-1} R_a^{A_j} + L_a^{A_j} + R_a^{B_j} + L_a^{B_j} \right) f \\
\{f, \tilde{t}_{\kappa_{\nu,n+2i}}\} &= p^a_{\kappa_{\nu,n+2i}}\left(R_a^{M_\nu} + L_a^{M_\nu} + L_a^{A_i} + R_a^{B_i} + L_a^{B_i}\right) f + \left(p^a_{\kappa_{\nu,n+2i}} - p^a_{m_\nu \circ \kappa_{\nu,n+2i} \circ m_\nu^{-1}}\right) R_a^{A_i} f \\
&\quad + \left(p^a_{\kappa_{\nu,n+2i}} - p^a_{m_\nu \circ \kappa_{\nu,n+2i} \circ m_\nu^{-1}}\right) \left(\sum_{j=\nu+1}^{n} R_a^{M_j} + L_a^{M_j} + \sum_{j=1}^{i-1} R_a^{A_j} + L_a^{A_j} + R_a^{B_j} + L_a^{B_j} \right) f,
\end{aligned} \tag{137}
$$

where we denote by $p^a : H \to \mathbb{R}$, $a = 1, \ldots, \dim \mathfrak{h}$, the maps $p^a : u = \mathrm{e}^{k^b J_b} \mapsto k^a$ and set $p^a_\lambda = p^a \circ \rho_\lambda$.

By using the graphical representation defined in section 3.2, we can determine the intersection points of these curves with representatives of the generators m_i, a_j, b_j and assign them between the different factors in the expression (136). Using this assignment of intersection points, we can derive the one-parameter groups of transformations associated with the brackets (137):

$$
T^t_{\tilde{t},a_i} : B_i \mapsto B_i {A_i}^t \tag{138}
$$

$$
T_{\tilde{t},\delta_i} : A_i \mapsto A_i H^t_{\delta_i} = B_i^{-1} A_i \tag{139}
$$

$$
\begin{aligned}
T^t_{\tilde{t},\alpha_i} : A_i &\mapsto A_i H^t_{\alpha_i} \\
B_{i-1} &\mapsto H^{-t}_{\alpha_i} B_{i-1} H^t_{\alpha_i} \\
A_{i-1} &\mapsto H^{-t}_{\alpha_i} A_{i-1}
\end{aligned} \tag{140}
$$

$$
\begin{aligned}
T^t_{\tilde{t},\epsilon_i} : A_i &\mapsto A_i H^t_{\epsilon_i} \\
A_k &\mapsto H^{-t}_{\epsilon_i} A_k H^t_{\epsilon_i} \\
B_k &\mapsto H^{-t}_{\epsilon_i} B_k H^t_{\epsilon_i} \qquad \forall 1 \leqslant k < i
\end{aligned} \tag{141}
$$

$$
\begin{aligned}
T^t_{\tilde{t},\kappa_{\nu,\mu}} : M_\nu &\mapsto H^{-t}_{\kappa_{\nu,\mu}} M_\nu H^t_{\kappa_{\nu,\mu}} \\
M_\mu &\mapsto H^{-t}_{\kappa_{\nu,\mu}} M_\mu H^t_{\kappa_{\nu,\mu}} \\
M_j &\mapsto \left[H^{-t}_{\kappa_{\nu,\mu}}, M_\nu\right] M_j \left[M_\nu, H^{-t}_{\kappa_{\nu,\mu}}\right] \qquad \forall \nu < j < \mu
\end{aligned} \tag{142}
$$

$$
\begin{aligned}
T^t_{\tilde{t},\kappa_{\nu,n+2i-1}} : M_\nu &\mapsto H^{-t}_{\kappa_{\nu,n+2i-1}} M_\nu H^t_{\kappa_{\nu,n+2i-1}} \\
A_i &\mapsto A_i H^t_{\kappa_{\nu,n+2i-1}} \\
X &\mapsto \left[H^{-t}_{\kappa_{\nu,n+2i-1}}, M_\nu\right] X \left[M_\nu, H^{-t}_{\kappa_{\nu,n+2i-1}}\right] \qquad X \in \{M_{\nu+1}, \ldots, M_n, A_1, \ldots, B_{i-1}\}
\end{aligned} \tag{143}
$$

$$
\begin{aligned}
T^t_{\tilde{t},\kappa_{\nu,n+2i}} : M_\nu &\mapsto H^{-t}_{\kappa_{\nu,n+2i}} M_\nu H^t_{\kappa_{\nu,n+2i}} \\
B_i &\mapsto H^{-t}_{\kappa_{\nu,n+2i}} B_i H^t_{\kappa_{\nu,n+2i}} \\
A_i &\mapsto H^{-t}_{\kappa_{\nu,n+2i}} A_i \left[M_\nu, H^{-t}_{\kappa_{\nu,n+2i}}\right] \\
X &\mapsto \left[H^{-t}_{\kappa_{\nu,n+2i}}, M_\nu\right] X [M_\nu, H^{-t}_{\kappa_{\nu,n+2i}}] \qquad X \in \{M_{\nu+1}, \ldots, M_n, A_1, \ldots B_{i-1}\},
\end{aligned} \tag{144}
$$

where we write $X^t := \mathrm{e}^{tp^a J_a}$ for $X = \mathrm{e}^{p^a J_a} \in H$ and set $H_\lambda = \rho_\lambda(M_1, \ldots, B_g)$. By comparing expressions (138)–(144) with formulae (A.2)–(A.8) for the action of the Dehn twists on the generators of the fundamental group $\pi_1(S_{g,n}\setminus D)$, we see that these expressions agree if we set $t = 1$ and replace m, a, b in (A.2)–(A.8) with the corresponding capital letters. Hence, the identity (134) holds for the generating Dehn twists (A.2)–(A.8).

2. As corollary 5.5 implies that the transformations $T^t_{f,\lambda} : H^{n+2g} \to H^{n+2g}$ act by Poisson isomorphisms, the same holds for the generating Dehn twists, the transformations (138)–(144) and hence for all elements of the pure mapping class group $\mathrm{PMap}(S_{g,n}\setminus D)$. To prove the invariance of the Poisson structure under the full mapping class group $\mathrm{Map}(S_{g,n}\setminus D)$, it is therefore sufficient to demonstrate that (135) holds for the transformations on H^{n+2g} induced by the transformations (A.9) which permute the punctures. This can be verified by direct computation using the expressions (79), (80), (81) for the Poisson bracket. For a proof of an analogous statement for the (2+1)-dimensional Poincaré group and gauge groups of the form $G \ltimes \mathfrak{g}^*$, see also [4, 15].

3. To prove that identity (134) holds for any embedded curve, we make use of the fact that the mapping class group $\mathrm{Map}(S_{g,n}\setminus D)$ acts by Poisson isomorphisms. We consider an embedded curve $\lambda \in \pi_1(S_{g,n}\setminus D)$ and an embedded curve $\eta \in \pi_1(S_{g,n}\setminus D)$ obtained from λ via the action of an element $\varphi \in \mathrm{Map}(S_{g,n}\setminus D)$ on $\pi_1(S_{g,n}\setminus D)$

$$\eta = \varphi(\lambda) \qquad \varphi \in \mathrm{Map}(S_{g,n}\setminus D). \tag{145}$$

From the geometrical definition of the Dehn twists it follows that the action $D_\eta : H^{n+2g} \to H^{n+2g}$ of the Dehn twist along η is given by

$$D_\eta = \Phi_\varphi^{-1} \circ D_\lambda \circ \Phi_\varphi, \tag{146}$$

where $\Phi_\varphi : H^{n+2g} \to H^{n+2g}$ is the diffeomorphism induced by φ. On the other hand, the definition (75) of the maps $\rho_\lambda : H^{n+2g} \to H$ implies

$$\rho_{\varphi(\lambda)} = \rho_\lambda \circ \Phi_\varphi. \tag{147}$$

Using identity (111) and the fact that the diffeomorphism Φ_φ is a Poisson isomorphism, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f \circ \Phi_\varphi \circ T^t_{\tilde{t},\eta} &= \{f \circ \Phi_\varphi, \tilde{t}_\eta\} = \{f \circ \Phi_\varphi, \tilde{t}_\lambda \circ \Phi_\varphi\} = \{f, \tilde{t}_\lambda\} \circ \Phi_\varphi \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f \circ T^t_{\tilde{t},\lambda} \circ \Phi_\varphi \end{aligned} \tag{148}$$

and therefore

$$T^t_{\tilde{t},\eta} = \Phi_\varphi^{-1} \circ T^t_{\tilde{t},\lambda} \circ \Phi_\varphi. \tag{149}$$

By setting $t = 1$ and combining this identity with the corresponding identity (146) for the Dehn twist around η, we find that identity (134) holds for η if and only if it holds for λ.

4. Hence, it is sufficient to prove (134) for one element in each orbit of the action of $\mathrm{Map}(S_{g,n}\setminus D)$ on $\pi_1(S_{g,n}\setminus D)$. A set of curves $\lambda \in \pi_1(S_{g,n}\setminus D)$ spanning the orbits of the action of $\mathrm{Map}(S_{g,n}\setminus D)$ can be obtained using results from geometric topology. It has been shown, see for instance lemma 2.3.A. in [19], that the equivalence classes of all non-separating curves γ on the surface $S_{g,n}\setminus D$ are in the same orbit under the action of the mapping class group. We can apply the same argument to separating curves if we keep in mind that the punctures of the surface $S_{g,n}\setminus D$ can be distinguished via the conjugacy classes assigned to them. Hence, any two separating curves γ, γ' on $S_{g,n}\setminus D$ such that the two components of $(S_{g,n}\setminus D)\setminus\gamma$ and

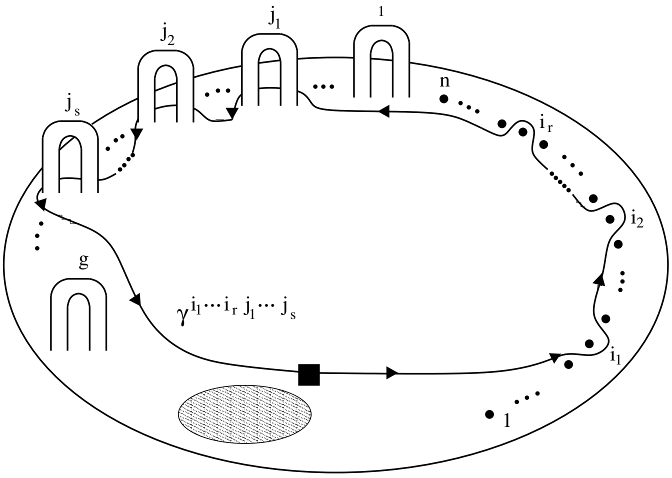

Figure 11. Separating curves on the surface $S_{g,n}\backslash D$.

$(S_{g,n}\backslash D)\backslash\gamma'$ contain the same number of handles and the same sets of punctures are in the same orbit under the action of the mapping class group. We therefore have to prove (134) for a single non-separating curve, such as the generators $a_i, \delta_i, \alpha_i, \epsilon_i, \kappa_{\nu,n+2i-1}$ or $\kappa_{\nu,n+2i}$ in (A.1) and the separating curves $\gamma^{i_1\ldots i_r j_1\ldots j_s}$ show in figure 11:

$$\begin{aligned}\gamma^{i_1\ldots i_r j_1\ldots j_s} &= h_{j_s}\circ h_{j_{s-1}}\cdots h_{j_2}\circ h_{j_1}\circ m_{i_r}\circ m_{i_{r-1}}\cdots m_{i_2}\circ m_{i_1}\\ &= \left(\overline{m}_1^{-1}\cdots\overline{m}_{i_1-1}^{-1}\right)\circ\left(\overline{m}_{i_1}^{-1}\cdots\overline{h}_{j_s}^{-1}\right)\circ\left(\overline{h}_{j_s-1}\cdots\overline{h}_{j_{s-1}+1}\right)\circ\ldots\circ\left(\overline{h}_{j_2-1}\cdots\overline{h}_{j_1+1}\right)\\ &\quad\circ\left(\overline{m}_{i_r-1}\cdots\overline{m}_{i_{r-1}+1}\right)\circ\ldots\circ\left(\overline{m}_{i_2-1}\cdots\overline{m}_{i_1+1}\right)\circ\left(\overline{m}_{i_1-1}\cdots\overline{m}_1\right)\\ &1\leqslant j_1<j_2<\cdots<j_s\leqslant j_{s+1}:=g,\ 1\leqslant i_1<i_2<\cdots<i_r\leqslant i_{r+1}:=n.\end{aligned}\tag{150}$$

Using formula (98) or (101), we can determine the Poisson bracket of a general function $f\in\mathcal{C}^\infty(H^{n+2g})$ with the Wilson loop observable $\tilde{t}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}$ and obtain

$$\begin{aligned}\{f,\tilde{t}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}\} &= p^a_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}\left(\sum_{k=1}^{r}R_a^{M_{i_k}}+L_a^{M_{i_k}}+\sum_{k=1}^{s}R_a^{A_{j_k}}+L_a^{A_{j_k}}+R_a^{B_{j_k}}+L_a^{B_{j_k}}\right)f\\ &\quad+\sum_{k=1}^{r}\left(p^a_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}-p^a_{(m_{i_k}\cdots m_{i_1})\circ\gamma^{i_1\ldots i_r j_1\ldots j_s}\circ(m_{i_k}\cdots m_{i_1})^{-1}}\right)\sum_{i_k<j<i_{k+1}}\left(R_a^{M_j}+L_a^{M_j}\right)f\\ &\quad+\sum_{k=0}^{s-1}\left(p^a_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}-p^a_{(h_{j_k}\cdots h_{j_1}m_{i_r}\cdots m_{i_1})\circ\gamma^{i_1\ldots i_r j_1\ldots j_s}\circ(h_{j_k}\cdots h_{j_1}m_{i_r}\cdots m_{i_1})^{-1}}\right)\\ &\quad\times\sum_{j_k<j<j_{k+1}}\left(R_a^{A_j}+L_a^{A_j}+R_a^{B_j}+L_a^{B_j}\right)f\end{aligned}\tag{151}$$

where we set $i_{r+1}=n, j_0=0, h_{j_0}=1$. Using the graphical representation of $\gamma^{i_1\ldots i_r j_1\ldots j_s}$ defined in section 3.2, we can determine the associated one-parameter group of diffeomorphisms:

$$
\begin{aligned}
T^t_{\tilde{t},\gamma_{i_1\ldots i_r j_1\ldots j_s}} : M_{i_k} &\mapsto H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}} M_{i_k} H^t_{\gamma_{i_1\ldots i_r j_1\ldots j_s}} && k=1,\ldots,r\\
A_{j_k} &\mapsto H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}} A_{j_k} H^t_{\gamma_{i_1\ldots i_r j_1\ldots j_s}} && k=1,\ldots,s\\
B_{j_k} &\mapsto H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}} B_{j_k} H^t_{\gamma_{i_1\ldots i_r j_1\ldots j_s}} && k=1,\ldots,s\\
M_j &\mapsto \left[H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}}, M_{i_l}\cdots M_{i_1}\right] M_j \left[M_{i_l}\cdots M_{i_1}, H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}}\right] && i_l < j < i_{l+1}\\
M_j &\mapsto \left[H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}}, M_{i_r}\cdots M_{i_1}\right] M_j \left[M_{i_r}\cdots M_{i_1}, H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}}\right] && i_r < j \leqslant n\\
A_j &\mapsto \left[H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}}, M_{i_r}\cdots M_{i_1}\right] A_j \left[M_{i_r}\cdots M_{i_1}, H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}}\right] && 1 \leqslant j < j_1\\
B_j &\mapsto \left[H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}}, M_{i_r}\cdots M_{i_1}\right] B_j \left[M_{i_r}\cdots M_{i_1}, H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}}\right] && 1 \leqslant< j < j_1\\
A_j &\mapsto \left[H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}}, H_{j_l}\cdots M_{i_1}\right] A_j \left[H_{j_l}\cdots M_{i_1}, H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}}\right] && j_l < j < j_{l+1}\\
B_j &\mapsto \left[H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}}, H_{j_l}\cdots M_{i_1}\right] B_j \left[H_{j_l}\cdots M_{i_1}, H^{-t}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}}\right] && j_l < j < j_{l+1}
\end{aligned}
\tag{152}
$$

where we write $H_\lambda = \rho_\lambda(M_1, \ldots, B_g)$ and for $X = \mathrm{e}^{p^a J_a} \in H$, $X^t := \mathrm{e}^{t p^a_\lambda J^a}$. The action of the Dehn twists around $\gamma_{i_1\ldots i_r j_1\ldots j_s}$ on the generators of the fundamental group $\pi_1(S_{g,n}\backslash D)$ is given by

$$
\begin{aligned}
\mathrm{d}_{\gamma_{i_1\ldots i_r j_1\ldots j_s}} : m_{i_k} &\mapsto \gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s} m_{i_k} \gamma_{i_1\ldots i_r j_1\ldots j_s} && k=1,\ldots,r\\
a_{j_k} &\mapsto \gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s} a_{j_k} \gamma_{i_1\ldots i_r j_1\ldots j_s} && k=1,\ldots,s\\
b_{j_k} &\mapsto \gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s} b_{j_k} \gamma_{i_1\ldots i_r j_1\ldots j_s} && k=1,\ldots,s
\end{aligned}
\tag{153}
$$

$$
\begin{aligned}
m_j &\mapsto \left[\gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s}, m_{i_l}\cdots m_{i_1}\right] m_j \left[\gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s}, m_{i_l}\cdots m_{i_1}\right]^{-1} && i_l < j < i_{l+1}\\
m_j &\mapsto \left[\gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s}, m_{i_r}\cdots m_{i_1}\right] m_j \left[\gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s}, m_{i_r}\cdots m_{i_1}\right]^{-1} && i_r < j \leqslant n\\
a_j &\mapsto \left[\gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s}, m_{i_r}\cdots m_{i_1}\right] a_j \left[\gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s}, m_{i_r}\cdots m_{i_1}\right]^{-1} && 1 \leqslant j < j_1\\
b_j &\mapsto \left[\gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s}, m_{i_r}\cdots m_{i_1}\right] b_j \left[\gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s}, m_{i_r}\cdots m_{i_1}\right]^{-1} && 1 \leqslant j < j_1\\
a_j &\mapsto \left[\gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s}, h_{j_l}\cdots m_{i_1}\right] a_j \left[\gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s}, h_{j_l}\cdots m_{i_1}\right]^{-1} && j_l < j < j_{l+1}\\
b_j &\mapsto \left[\gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s}, h_{j_l}\cdots m_{i_1}\right] b_j \left[\gamma^{-1}_{i_1\ldots i_r j_1\ldots j_s}, h_{j_l}\cdots m_{i_1}\right]^{-1} && j_l < j < j_{l+1},
\end{aligned}
\tag{153}
$$

and comparing with expression (152) we find that the diffeomorphism of H^{n+2g} induced by the Dehn twist (153) agrees with the transformation (152) for $t = 1$. Hence, equation (134) holds for the separating curves $\gamma^{i_1\ldots i_r j_1\ldots j_s}$ and therefore for all embedded curves $\lambda \in \pi_1(S_{g,n}\backslash D)$. □

The Wilson loop observables $\tilde{t}_\lambda$ associated with the bilinear form $\langle,\rangle$ in the Chern–Simons action therefore act as the Hamiltonians for infinitesimal Dehn twists along embedded curves $\lambda \in \pi_1(S_{g,n}\backslash D)$. While the number and form of other Ad-invariant functions $f \in \mathcal{C}^\infty(H)$ is a property of the gauge group H, the observables $\tilde{t}_\lambda$ are present generically in Chern–Simons theory and the associated transformations have a geometrical interpretation. The fact that Dehn twists are infinitesimally generated via the Poisson bracket allows us to determine the transformation behaviour of the Poisson bracket and the transformations $T^t_{f,\lambda}$ under a general automorphism of the fundamental group $\pi_1(S_{g,n}\backslash D)$ which arises from a homeomorphism of $S_{g,n}\backslash D$. We have the following theorem.

Theorem 6.3. *Let $\varphi \in Aut(\pi_1(S_{g,n}\backslash D))$ be an automorphism of the fundamental group which satisfies the requirement (4) with $w = 1$, $\varphi(m_D) = m_D^{\pm 1}$, and denote by $\Phi_\varphi : H^{n+2g} \to H^{n+2g}$ the induced diffeomorphism of H^{n+2g}.*

(1) Then, the transformation of the Fock–Rosly Poisson bracket $\{,\}_r$ *associated with the classical r-matrix r under* Φ_φ *is given by*

$$\{f\circ\Phi_\varphi, f\circ\Phi_\varphi\}_r = \begin{cases}\{f,g\}_r\circ\Phi_\varphi & \textit{if}\quad \varphi(m_D)=m_D\\ \{f,g\}_{-\sigma(r)}\circ\Phi_\varphi & \textit{if}\quad \varphi(m_D)=m_D^{-1}\end{cases}\quad \forall f,g\in\mathcal{C}^\infty(H^{n+2g}). \tag{154}$$

(2) For any conjugation invariant $f\in\mathcal{C}^\infty(H)$ *and any* $\lambda\in\pi_1(S_{g,n}\backslash D)$ *with an embedded representative, the flow* $T^t_{f,\lambda}$ *generated by the observable* f_λ *satisfies*

$$T^t_{f,\varphi(\lambda)} = \begin{cases}\Phi_\varphi^{-1}\circ T^t_{f,\lambda}\circ\Phi_\varphi & \textit{if}\quad \varphi(m_D)=m_D\\ \Phi_\varphi^{-1}\circ T^{-t}_{f,\lambda}\circ\Phi_\varphi & \textit{if}\quad \varphi(m_D)=m_D^{-1}.\end{cases} \tag{155}$$

Proof. To prove identities (154), (155) we note that automorphisms $\varphi\in\mathrm{Aut}(\pi_1(S_{g,n}\backslash D))$ satisfying the conditions (4) with $\varphi(m_D)=m_D$ correspond to elements of the mapping class group $\mathrm{Map}(S_{g,n}\backslash D)$, while automorphisms with $\varphi(m_D)=m_D^{-1}$ are obtained by composing elements of the mapping class group $\mathrm{Map}(S_{g,n}\backslash D)$ with the involution I:

$$\varphi(m_D)=m_D^{-1}\quad\Rightarrow\quad \exists\psi\in\mathrm{Map}(S_{g,n}\backslash D):\varphi=\psi\circ I. \tag{156}$$

In the first case, identities (154), (155) follow from the proof of theorem 6.2. For automorphisms of the form (156), we note that the diffeomorphisms $\Phi_\eta,\Phi_\tau: H^{n+2g}\to H^{n+2g}$ associated with arbitrary $\eta,\tau\in\mathrm{Aut}(\pi_1(S_{g,n}\backslash D))$ satisfy $\Phi_{\tau\circ\eta}=\Phi_\eta\circ\Phi_\tau$. Recalling identity (94) from theorem 4.3 which specifies the transformation of the Poisson bracket under the diffeomorphism Φ_I associated with the involution, we then find

$$\begin{aligned}\{f\circ\Phi_{\psi\circ I}, g\circ\Phi_{\psi\circ I}\}_r &= \{f\circ\Phi_I, g\circ\Phi_I\}_r\circ\Phi_\psi = \{f,g\}_{-\sigma(r)}\circ\Phi_I\circ\Phi_\psi\\ &= \{f,g\}_{-\sigma(r)}\circ\Phi_{\psi\circ I}.\end{aligned} \tag{157}$$

To prove identity (155), we apply (154) to the bracket of the observable $f_{I(\lambda)}$ with a general function $g\in\mathcal{C}^\infty(H^{n+2g})$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g\circ T^t_{f,I(\lambda)} &= \{f_{I(\lambda)},g\} = \{f_\lambda\circ\Phi_I, g\} = -\{f_\lambda, g\circ\Phi_I\}\circ\Phi_I\\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g\circ\Phi_I\circ T^t_{f,\lambda}\circ\Phi_I \quad\Rightarrow\quad T^t_{f,I(\lambda)}=\Phi_I\circ T^{-t}_{f,\lambda}\circ\Phi_I\\ &\quad \forall f\in\mathcal{C}^\infty(H).\end{aligned} \tag{158}$$

Using the identity $\Phi_{\psi\circ I}=\Phi_I\circ\Phi_\psi$ together with (157) then proves the claim. □

7. Outlook and conclusions

In this paper, we defined the dual of a set of generators of the fundamental groups $\pi_1(S_{g,n})$, $\pi_1(S_{g,n}\backslash D)$ and applied it to the moduli space of flat connections on $S_{g,n}$. This dual is given by an involution of the fundamental group $\pi_1(S_{g,n}\backslash D)$ and can be viewed as a dual graph for the set of curves representing these generators. In particular, the expression of the homotopy equivalence class of a general embedded curve on $S_{g,n}\backslash D$ in terms of the dual generators determines both the number of its intersection points with the representatives of the generators and the order in which the intersection points occur on these representatives.

By applying this involution to Fock and Rosly's description [3] of the moduli space of flat H-connections, we showed that the Poisson structure takes a particularly simple form when expressed in terms of both the holonomies along the original generators and those along their duals. This allowed us to give explicit expressions for the Poisson brackets of the Wilson loop observables associated with general conjugation-invariant functions on H and general embedded curves on $S_{g,n}\setminus D$ and to derive the associated flows on phase space. For the generic observables constructed from the non-degenerate Ad-invariant bilinear form in the Chern–Simons action, we showed that the associated flows represent infinitesimal Dehn twists. Using the fact that these Dehn twists are generated via the Poisson bracket, we determined the transformation of the Poisson structure under general automorphism of $\pi_1(S_{g,n}\setminus D)$ induced by a homeomorphisms of $S_{g,n}\setminus D$.

The results in this paper generalize Goldman's classic results on twist flows [11] to surfaces with punctures. However, in contrast to the formulation in [11], our description of these flows is purely algebraic and gives concrete expressions for their action on the holonomies along a set of generators of the fundamental group $\pi_1(S_{g,n}\setminus D)$. As the holonomies along these generators appear as the basic variables in many approaches to the quantization of Chern–Simons theory [6–10], this could prove useful in applying the results to the associated quantum theories.

In the case of infinitesimal Dehn twists, the associated quantum transformations have been determined for some gauge groups. For Chern–Simons theory with compact semisimple gauge groups, the quantum action of the Dehn twists has been derived by Alekseev, Grosse and Schomerus [6–8] who found that it is implemented via the ribbon element of the quantum group arising in the combinatorial quantization procedure. The case of semidirect product gauge groups $G \ltimes \mathfrak{g}^*$ is investigated in [15], where it is shown that the mapping class group acts on the representation spaces of the quantum algebra and that the action of Dehn twists is given by the ribbon element of the quantum double $D(G)$. This raises the question if the implementation of Dehn twists in the quantum theory via the ribbon element of an associated quantum group is also present for other gauge groups such as the group $SL(2,\mathbb{C})$ investigated in [9]. It would also be interesting to use the results of this paper to determine the quantum action of the flows generated by other Wilson loop observables and to see how these flows reflect the properties of the quantum group arising in the quantization of the theory.

Finally, Wilson loop observables and the transformations these observables generate via the Poisson bracket play an important role in the Chern–Simons formulation of (2+1)-dimensional gravity. For (2+1)-dimensional gravity with vanishing cosmological constant, it is shown in [5] that the flows generated by a particular set of Wilson loop observables correspond to the construction of evolving (2+1)-spacetimes via grafting. The results of this paper allow one to generalize these results to other values of the cosmological constant, for which the Poisson structure is more involved, and to establish a common framework which treats the cosmological constant as a deformation parameter [12].

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Appendix A. The generators of the mapping class group

In this appendix, we give a set of generators of the mapping class groups $\mathrm{Map}(S_{g,n}\setminus D)$ and $\mathrm{Map}(S_{g,n})$ and provide explicit expressions for their action on a set of generators of the fundamental groups $\pi_1(S_{g,n}\setminus D)$, $\pi_1(S_{g,n})$. A set of generators and relations of the mapping

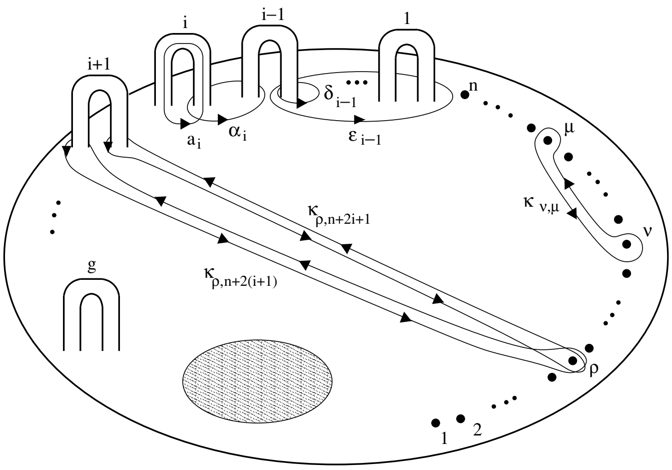

Figure 12. The curves associated with the generators of the mapping class group PMap($S_{g,n}\backslash D$).

class group on oriented 2-surfaces was first derived by Birman [16, 17]. In this paper we work with the set used in [8], which was first presented in [20], see also [21].

Both the pure mapping class groups PMap($S_{g,n}\backslash D$) and PMap($S_{g,n}$) are generated by a finite set of Dehn twists around certain curves on the surfaces $S_{g,n}\backslash D$, $S_{g,n}$, but the latter with additional relations. These curves are depicted in figure 12, and in terms of the generators m_i, a_j, b_j of $\pi_1(S_{g,n}\backslash D)$, $\pi_1(S_{g,n})$ their homotopy equivalence classes are given by

$$\begin{aligned}
&a_i \qquad i = 1, \ldots, g\\
&\delta_i = a_i^{-1} \circ b_i^{-1} \circ a_i \qquad i = 1, \ldots, g\\
&\alpha_i = a_i^{-1} \circ b_i^{-1} \circ a_i \circ b_{i-1}\\
&\epsilon_i = a_i^{-1} \circ b_i^{-1} \circ a_i \circ h_{i-1} \cdots h_1 = \qquad i = 2, \ldots, g\\
&\kappa_{\nu,\mu} = m_\mu \circ m_\nu \qquad 1 \leqslant \nu < \mu \leqslant n\\
&\kappa_{\nu,n+2i-1} = a_i^{-1} \circ b_i^{-1} \circ a_i \circ m_\nu\\
&\kappa_{\nu,n+2i} = b_i \circ m_\nu.
\end{aligned} \tag{A.1}$$

Dehn twists around embedded curves representing the elements (A.1) induce automorphisms of the fundamental groups $\pi_1(S_{g,n}\backslash D)$, $\pi_1(S_{g,n})$ which are given by their action on the generators m_i, a_j, b_j. In the following, we list only those generators which do not transform trivially:

$$\mathrm{d}_{a_i} : b_i \mapsto b_i a_i \tag{A.2}$$

$$\mathrm{d}_{\delta_i} : a_i \mapsto a_i \delta_i = b_i^{-1} a_i \tag{A.3}$$

$$\begin{aligned}
\mathrm{d}_{\alpha_i} : a_i &\mapsto b_i^{-1} a_i b_{i-1} = a_i \alpha_i\\
b_{i-1} &\mapsto \alpha_i^{-1} b_{i-1} \alpha_i\\
a_{i-1} &\mapsto \alpha_i^{-1} a_{i-1}
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\mathrm{d}_{\epsilon_i} : a_i &\mapsto b_i^{-1} a_i k_{i-1} \ldots k_1 = a_i \epsilon_i\\
a_k &\mapsto \epsilon_i^{-1} a_k \epsilon_i\\
b_k &\mapsto \epsilon_i^{-1} b_k \epsilon_i \qquad \forall 1 \leqslant k < i
\end{aligned} \tag{A.5}$$

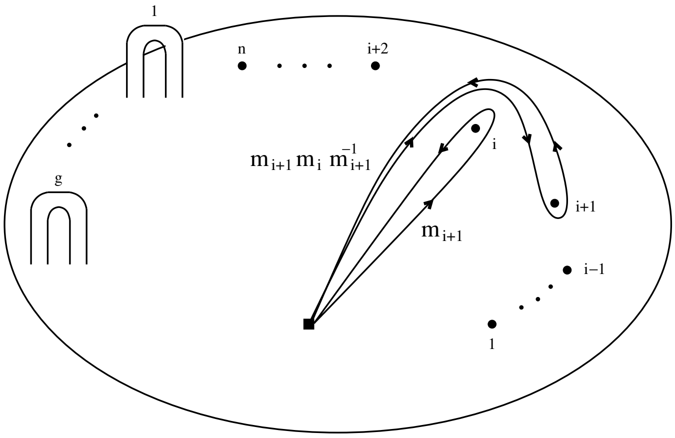

Figure 13. The generators of the braid group on the surface $S_{g,n} \backslash D$.

$$\begin{aligned} \mathrm{d}_{\kappa_{\nu,\mu}} : m_\nu &\mapsto \kappa_{\nu,\mu}^{-1} m_\nu \kappa_{\nu,\mu} \\ m_\mu &\mapsto \kappa_{\nu,\mu}^{-1} m_\mu \kappa_{\nu,\mu} \\ m_j &\mapsto \left[\kappa_{\nu,\mu}^{-1}, m_\nu\right] m_j \left[m_\nu, \kappa_{\nu,\mu}^{-1}\right] \qquad \forall \nu 1 j < \mu \end{aligned} \tag{A.6}$$

$$\begin{aligned} \mathrm{d}_{\kappa_{\nu,n+2i-1}} : m_\nu &\mapsto \kappa_{\nu,n+2i-1}^{-1} m_\nu \kappa_{\nu,n+2i-1} \\ a_i &\mapsto a_i \kappa_{\nu,n+2i-1} \\ x &\mapsto [\kappa_{\nu,n+2i-1}^{-1}, m_\nu] x [m_\nu, \kappa_{\nu,n+2i-1}^{-1}] \qquad x \in \{m_{\nu+1}, \ldots, m_n, a_1, \ldots, b_{i-1}\} \end{aligned} \tag{A.7}$$

$$\begin{aligned} \mathrm{d}_{\kappa_{\nu,n+2i}} : m_\nu &\mapsto \kappa_{\nu,n+2i}^{-1} m_\nu \kappa_{\nu,n+2i} \\ b_i &\mapsto \kappa_{\nu,n+2i}^{-1} b_i \kappa_{\nu,n+2i} \\ a_i &\mapsto \kappa_{\nu,n+2i}^{-1} a_i [m_\nu, \kappa_{\nu,n+2i}^{-1}] \\ x &\mapsto [\kappa_{\nu,n+2i}^{-1}, m_\nu] x [m_\nu, \kappa_{\nu,n+2i}^{-1}] \qquad x \in \{m_{\nu+1}, \ldots, m_n, a_1, \ldots b_{i-1}\}. \end{aligned} \tag{A.8}$$

A set of generators of the full mapping class group of the surface $S_{g,n} \backslash D$ is obtained by supplementing this set of generators with the generators σ^i, $i = 1, \ldots, n$, of the braid group. The action of these generators on the loops m_i around the punctures is shown in figure 13. They leave invariant all generators of the fundamental group except m_i and m_{i+1}, on which they act according to

$$\sigma^i : m_i \mapsto m_{i+1} \qquad m_{i+1} \mapsto m_{i+1} m_i m_{i+1}^{-1}. \tag{A.9}$$

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